

# PhD Qualification Exam in Applied Mathematics

Spring, 2026

1. (10 points) Assume that locally the function  $f \in C^2(\mathbb{R})$  has a unique zero  $x_*$  with  $f'(x_*) \neq 0$ . Consider the iteration

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)}, k = 0, 1, \dots$$

Prove that  $x_k \rightarrow x_*$ , and estimate its order of local convergence.

2. (20 points) For any continuous function  $f$ , define  $\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$  and  $\|f\|_2 = \sqrt{\langle f, f \rangle}$ . Write  $a_n = \frac{(2n)!}{2^n(n!)^2}$ .

Define the  $n$ -th monic Legendre polynomial by:  $P_0(x) = 1, P_1(x) = x, P_{n+1}(x) = xP_n(x) - \frac{n^2}{4n^2 - 1}P_{n-1}(x)$ .

- (a) Compute  $\langle P_n, P_m \rangle$ , for  $m, n \in \mathbb{N}$ .  
 (b) Prove that  $P_n$  solves  $\min_{\substack{\text{polynomial } p \\ \deg p \leq n, p^{(n)}(1) = n!}} \|p\|_2$  and compute the minimal value.  
 (c) For  $f \in C^{n+1}[-1, 1]$ , choose the interpolation nodes  $x_0, x_1, \dots, x_n$  as the  $n + 1$  zeros of  $P_{n+1}(x)$ , and denote its degree- $n$  interpolation polynomial by  $L_n$ . Prove

$$\|f - L_n\|_2 \leq \frac{2\|f^{(n+1)}\|_2}{(2n + 3)a_{n+1}(n + 1)!}.$$

- (d) For  $f \in C^{2n+2}[-1, 1]$ , derive the Gauss quadrature formula of  $I(f) = \int_{-1}^1 f(x) dx$  for  $n + 1$  quadrature nodes, written by  $I_{n+1}(f)$ , and estimate its error  $|I(f) - I_{n+1}(f)|$ .

3. (20 points) Given  $A \in \mathbb{R}^{l \times m}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{l \times p}$ . Consider the problem  $\min_X \|AXB - C\|_F$ , where  $\|\cdot\|_F$  is the Frobenius norm of a matrix.

- (a) Prove that the minimal value is 0, if and only if  $\text{rank}(A) = \text{rank} \left( \begin{bmatrix} A & C \end{bmatrix} \right), \text{rank}(B) = \text{rank} \left( \begin{bmatrix} B \\ C \end{bmatrix} \right)$ .  
 (b) Show that there exists a unique minimal point  $X_*$ , if and only if  $\text{rank}(A) = m, \text{rank}(B) = n$ .  
 (c) Propose a numerical method to compute the unique  $X_*$  in (b).  
 (d) Does  $X_*$  also minimize  $\|AXB - C\|_2$ ? Why? Here  $\|\cdot\|_2$  is the spectral norm of a matrix.

4. (15 points) For the advection equation  $u_t + u_x = 0$ , analyze the stability of the implicit schemes

(a)  $\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{u_j^{n+1} - u_{j-1}^{n+1}}{h} = 0;$

(b)  $\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{u_{j+1}^{n+1} - u_j^{n+1}}{h} = 0;$

(c)  $\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} = 0.$

5. (15 points) Write and prove the maximum principle of the centered finite difference scheme for discretizing the equation

$$u_{xx} + u_{yy} + du_x + eu_y + fu = 0, \quad f < 0,$$

under some suitable assumptions. Apply the maximum principle to show the stability of the backward Euler scheme for the parabolic equation

$$u_t - (u_{xx} + u_{yy} + du_x + eu_y + fu) = 0, \quad (x, y) \in (0, 1) \times (0, 1),$$

with homogeneous Dirichlet boundary conditions.

**You can choose either 6 or 7 to answer. The points will be decided as  $\max(6, 7)$ .**

6. (20 points) Consider the following system

$$\ddot{x} - 2x - x^2 + x^3 = 0.$$

- (a) (6 points) Find the static solutions to the system.  
(b) (14 points) For sufficiently small but nonzero  $\epsilon$ , show that the orbit starting at  $(x(0), \dot{x}(0)) = (2 + \epsilon, 0)$  is closed in the phase space, i.e.,  $x - \dot{x}$  plane. Find the approximation of the period of this orbit up to  $O(\epsilon^2)$ .

7. (20 points) The polar of  $C \subseteq \mathbf{R}^n$  is defined as the set

$$C^\circ = \{y \in \mathbf{R}^n \mid y^T x \leq 1 \text{ for all } x \in C\}.$$

- (a) (2 points) Show that  $C^\circ$  is convex (even if  $C$  is not).  
(b) (3 points) What is the polar of a cone?  
(c) (3 points) What is the polar of the unit ball for a norm  $\|\cdot\|$ ?  
(d) (4 points) What is the polar of the set  $C = \{x \mid \mathbf{1}^T x = 1, x \succeq 0\}$ ?  
(e) (8 points) Show that if  $C$  is closed and convex, with  $0 \in C$ , then  $(C^\circ)^\circ = C$ .