Qualifying Exam: 2025 Spring

考试课程: Probability & Statistics 姓名: ____________ 学号: ____________

- There are 11 problems in this exam (3 pages). You need to choose 8 of them to solve. If you select more than 8, only the first 8 that you have worked on will be graded. Note that 4 of the problems are worth 15 points each and the rest 10 points each.
- You must follow all the rules of exam taking. Misconducts will be subject to proper disciplinary actions by the Center.
- You must provide all necessary details for full credits. A final answer with no or little explanation/derivation, even if correct, receives a minimal credit.

1. (10 points) Let $\{U_n\}$ be i.i.d. with uniform distribution on $[0, 1]$. For $n \geq 2$, define
$$X_n := \min\{U_1, \ldots, U_n\}, \quad Y_n = \max\{U_1, \ldots, U_n\}.$$
   (1) Derive the joint distribution of (X_n, Y_n).
   (2) Calculate $\mathbb{E}[X_n \,|\, \sigma(Y_n)]$.

2. (10 points) Suppose (X, Y, Z) is three-dimensional mean-zero Gaussian vector with covariance matrix given by
$$\begin{pmatrix} 1 & \rho & \tau \\ \rho & 1 & 0 \\ \tau & 0 & 1 \end{pmatrix}.$$
   (1) Show that $\rho^2 + \tau^2 \leq 1$.
   (2) Show that there exist real numbers a, b, c and a standard Gaussian random variable W independent of (Y, Z) such that $X = aY + bZ + cW$ and derive a, b, c.

3. (15 points) Suppose $\{\xi_i\}$ are i.i.d. Bernoulli random variables with $\mathbb{P}[\xi_1 = 1] = \mathbb{P}[\xi_1 = -1] = 1/2$. Set $S_n = \sum_{j=1}^n \xi_j$ and define $\tau = \min\{n : S_n = -a \text{ or } b\}$ for $a, b \in \mathbb{Z}_{>0}$.
   (1) Calculate $\mathbb{E}\left[e^{\lambda S_n}\right]$ for $\lambda \in \mathbb{R}$.
   (2) Calculate $\mathbb{E}\left[s^{\tau}\right]$ for $s \in (0, 1)$.

4. (10 points) Suppose $\{\xi_i\}$ are i.i.d. Bernoulli random variables with $\mathbb{P}[\xi_1 = 1] = \mathbb{P}[\xi_1 = -1] = 1/2$. Set
$$X_n = \sum_{j=1}^n 2^{j-1} \xi_j.$$
   (1) Define $T = \min\{j \geq 1 : \xi_j = +1\}$. Derive the law of T and calculate $\mathbb{E}[T]$.
   (2) Show that $\{X_n\}$ is a martingale and calculate $\mathbb{E}[X_T]$.

5. (10 points) Let $\{B_t\}_{t \geq 0}$ be a one-dimensional Brownian motion starting from the origin (i.e, $B_0 = 0$). For $a \in (0, \infty)$, define $T_a := \inf\{t \geq 0 : |B_t| = a\}$.
   (1) Prove that $T_a < \infty$ a.s.

(2) Prove that T_a and $\mathbf{1}_{\{B_{T_a}=a\}}$ are independent.

6. (15 points) Suppose X and Y are i.i.d with $\mathbb{E}[X]=0$ and $\mathrm{var}(X)=1$. Suppose further that

$$\mathbb{E}[(X-Y)^2 \mid \sigma(X+Y)] = 2.$$

Show that X and Y are independent standard Gaussian random variables.

7. (10 points) Let $X_1, ..., X_n$ $(n \geq 2)$ be iid uniform $U(0, \theta)$ where $0 < \theta < \infty$.

(a) Under squared error loss $l(\theta, \delta(\boldsymbol{X})) = (\delta(\boldsymbol{X}) - \theta)^2$, derive the generalized Bayes rule for the improper prior

$$\pi(\theta) = 1,\ 0 < \theta < +\infty.$$

(b) Is the Bayes rule derived in (a) consistent for θ?

8. (10 points) Let $X_1, ..., X_n$ be iid Gamma$(2025, \theta)$ with $\theta > 0$ unknown. Recall that the probability density function of a Gamma(α, θ) is $f(x \mid \alpha, \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{\frac{-x}{\theta}}, x > 0$, where $\Gamma(\alpha) = (\alpha - 1)!$ for a positve integer α.

(a) Consider the hypothesis testing problem $H_0 : \theta = 1$ vs. $H_1 : \theta \neq 1$. Derive the UMP level $\alpha = 0.05$ test if it exists. If it doesn't exist, please argue why it does not exist.

(b) Derive the MLE for $\frac{1}{\theta}$, what is the asymptotic distribution of this MLE for $\frac{1}{\theta}$?

9. (15 points) Let $(X_1, Y_1), ..., (X_n, Y_n)$ be iid random bivariate vectors. Let ρ be the correlation coefficient between X_i and Y_i. Let

$$\hat{\rho} = \frac{1}{(n-1)\sqrt{S_X^2 S_Y^2}} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}),$$

where $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$, $\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$, $S_X^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$, $S_Y^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$.

(a) Assume that $\mathrm{E}|X_i|^4 < \infty$ and $\mathrm{E}|Y_i|^4 < \infty$, prove that $\sqrt{n}(\hat{\rho} - \rho) \longrightarrow \mathrm{N}(0, c)$ for some value c. Please identify c in terms of moments of (X_i, Y_i), however, there is no need to simplify your calculation on c.

(b) Further, assume that (X_i, Y_i) follows a bivariate normal distribution that

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right).$$

The probability density function of the above bivariate normal distribution is given by

$$f(x, y \mid \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[x^2 - 2\rho xy + y^2]\}.$$

Find a minimal sufficient statistic for ρ. Is this minimal sufficient statistic complete or not? Please explain.

10. (10 points) Let $X_1, ..., X_n$ be iid from $N(0, \sigma^2)$ with $\sigma > 0$ unknown. Let $W = \frac{1}{n} \sum_{i=1}^n |X_i|$. Recall χ_k^2 has probability density function

$$\frac{1}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, x \geq 0.$$

(a) Find an unbiased estimator of σ based on W and identify its limiting distribution.

(b) Is it admissible under the square error loss? Justify your answer.

11. (15 points) Let $X_1, ..., X_n$ be iid from exponential distribution with density $f(x; \lambda, \alpha) = \lambda e^{-\lambda(x-\alpha)} I_{\{x \geq \alpha\}}$, where $\lambda > 0$ and $-\infty < \alpha < \infty$ are unknown parameters.

    (a) Find the MLE of (λ, α), say $(\hat{\lambda}, \hat{\alpha})$.

    (b) What distribution does $\frac{\sqrt{n}(\hat{\lambda}-\lambda)}{n(\hat{\alpha}-\alpha)}$ converge to? You will receive partial credits if you can identify the distribution(s) of the numerator and/or denominator.