

# 清华大学考试试题专用纸

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Everyone needs to solve Problem 1, 2, 3, 4 (10 points each). Please select four problems in the following table and solve them (15 points each).

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|-----------|-----------|-----------|-----------|-----------|------------|
| Problem 5 | Problem 6 | Problem 7 | Problem 8 | Problem 9 | Problem 10 |
|           |           |           |           |           |            |

1. (10 points) Let

$$SO(3) = \{A \in M_3(\mathbb{R}) : A^T A = I, \det A = 1\}.$$

Compute  $H_k(SO(3); \mathbb{Z})$  for every  $k$ .

2. Let  $M \subset \mathbb{R}^n$  be a compact  $C^2$  embedded submanifold without boundary. Put

$$\nu(M) = \{(p, v) : p \in M, v \perp T_p M\}, \quad \nu_\epsilon(M) = \{(p, v) \in \nu(M) : \|v\| < \epsilon\},$$

and define  $\Phi(p, v) = p + v$ .

- (a) (5 points) Prove that there exists  $\epsilon > 0$  such that  $\Phi|_{\nu_\epsilon(M)}$  is a  $C^1$ -diffeomorphism onto

$$U_\epsilon(M) = \{x \in \mathbb{R}^n : \text{dist}(x, M) < \epsilon\}.$$

- (b) (5 points) Show that the  $C^2$  hypothesis cannot in general be weakened to  $C^1$ . More precisely, let  $N \subset \mathbb{R}^2$  be a compact  $C^1$  embedded circle which, in a neighborhood of the origin, is given by the arc

$$\gamma(t) = (t, |t|^{3/2}), \quad |t| \leq \delta,$$

You may assume without proof that such an embedded circle  $N$  exists. Prove that for every  $\epsilon > 0$ , the normal map

$$\Phi_N : \nu_\epsilon(N) \longrightarrow \mathbb{R}^2, \quad \Phi_N(p, v) = p + v,$$

is not injective.

3. (10 points) Let  $(M, g)$  be complete and let  $X$  be a Killing field. Prove that every connected component of

$$Z(X) = \{p \in M : X(p) = 0\}$$

is a closed submanifold of even codimension.

4. (10 points) Let  $M^n$  be oriented,  $n \geq 3$ , and let  $g$  and  $h$  be Riemannian metrics with Hodge-star operators  $*_g$  and  $*_h$ . Suppose

$$*_g \eta = *_h \eta \quad \text{for every smooth one-form } \eta.$$

Prove that  $g = h$  when  $n \geq 3$ . What happens when  $n = 2$ ?

5. Let

$$G = SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : |z|^2 + |w|^2 = 1 \right\}.$$

You may use that  $G$  is a compact, connected, smooth real 3-dimensional Lie group. Let

$$\iota: G \longrightarrow G, \quad \iota(A) = A^{-1},$$

be the inversion map.

(a) (5 points) Determine the Lie algebra  $\mathfrak{su}(2) = T_I G$  and prove that

$$d\iota_I = -\text{id}.$$

(b) (10 points) Let  $F: G \rightarrow G$  be a smooth map homotopic to  $\iota$ . Prove that  $F$  has at least one fixed point.

6. Let  $\nabla$  and  $\hat{\nabla}$  be torsion-free affine connections on an  $n$ -manifold, where  $n \geq 2$ . Assume that they have the same geodesic paths. In other words, assume that every geodesic for either connection becomes a geodesic for the other after a regular reparametrization.

(a) (7 points) Prove that there is a unique one-form  $\alpha$  such that

$$\hat{\nabla}_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X$$

for all vector fields  $X, Y$ .

(b) (8 points) Prove conversely that connections related by this formula have the same geodesic paths.

7. (15 points) Let  $X = \mathbb{RP}^2 \vee S^1$ , with the wedge point as basepoint. Let  $a$  be a loop in  $\mathbb{RP}^2$  representing the generator of  $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$ , and let  $b$  be the standard loop in  $S^1$ . Let  $f: X \rightarrow X$  be the based map that is the identity on  $\mathbb{RP}^2$  and sends  $b$  to  $a^{-1}b$ . Let

$$T_f = (X \times [0, 1]) / ((x, 1) \sim (f(x), 0))$$

be the mapping torus of  $f$ . Compute  $\pi_1(T_f)$  and  $H_k(T_f; \mathbb{Z})$  for every  $k$ .

8. (15 points) Let  $(M^n, g)$  be closed and satisfy  $\text{Ric} \geq 0$ . Assume that the first Betti number is maximal:

$$b_1(M) = n.$$

Prove that  $(M, g)$  is isometric to a flat torus  $\mathbb{R}^n / \Lambda$  for some lattice  $\Lambda \subset \mathbb{R}^n$ .

9. (15 points) Let  $(M^n, g)$  be compact and have strictly positive sectional curvature. Let  $N^a, P^b \subset M$  be closed embedded totally geodesic submanifolds. Prove that if  $a + b \geq n$ , then  $N \cap P \neq \emptyset$ .

10. Let

$$E = \{[X : Y : Z] \in \mathbb{CP}^2 : Y^2 Z = X^3 - XZ^2\}.$$

The set  $E$  is connected; you are not required to prove this fact.

- (a) (5 points) Prove that  $E$  is a compact, smooth real 2-dimensional submanifold of  $\mathbb{CP}^2$ .
- (b) (5 points) Equip  $E$  with its complex orientation. Define

$$a = \{[x : y : 1] \in E : x \in [-1, 0], y \in \mathbb{R}\}, \quad b = \{[x : y : 1] \in E : x \in [0, 1], y \in i\mathbb{R}\}.$$

Show that  $a$  and  $b$  are smoothly embedded circles intersecting transversely at  $p = [0 : 0 : 1]$ . Using  $y$  as a local complex coordinate at  $p$ , orient  $a$  and  $b$  so that their positive tangent directions at  $p$  correspond to  $-1$  and  $-i$ , respectively. Prove that, with these orientations,  $a \cdot b = 1$ .

- (c) (5 points) It is a fact, which you may use without proof, that  $E$  is diffeomorphic to the torus  $T^2$ . On the affine chart  $Z \neq 0$ , write  $x = X/Z$  and  $y = Y/Z$ , and set

$$\omega = \frac{dx}{y}.$$

You may use without proof that  $\omega$  extends to a smooth closed complex-valued 1-form on all of  $E$ . Define

$$\alpha = \operatorname{Re} \omega, \quad \beta = \operatorname{Im} \omega,$$

Determine the homology classes

$$\operatorname{PD}([\alpha]), \quad \operatorname{PD}([\beta])$$

in  $H_1(E; \mathbb{R})$  in terms of  $[a]$  and  $[b]$ . Here,  $\operatorname{PD}$  denotes the Poincaré dual.

*Hint.* You may need the following integral:

$$\kappa = 2 \int_{-1}^0 \frac{dx}{\sqrt{x^3 - x}} = 2 \int_0^1 \frac{dx}{\sqrt{x - x^3}} = \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}.$$