QUALIFY EXAM FOR APPLIED MATHEMATICS

(1) (15 points) Suppose $k \leqslant n \leqslant m$, and $A \in \mathbb{C}^{m\times n}$.
- find a rank-k matrix X satsifying $\|A - X\|_2 \leqslant \|A - B\|_2$ for any rank-k matrix B. Here $\|\cdot\|_2$ is the spectral norm.
- find a rank-k matrix X satsifying $\|A - X\|_{\mathrm{F}} \leqslant \|A - B\|_{\mathrm{F}}$ for any rank-k matrix B. Here $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm.

(2) (20 points) Consider the Gauss-Seidel iteration for solving the linear equation $Ax = b$ for $A = [a_{ij}] \in \mathbb{R}^{n\times n}, b = [b_i] \in \mathbb{R}^n$ with $a_{ii} \neq 0$ for $i = 1, \dots, n$.

determine $x^{(0)} \in \mathbb{R}^n$.
for $k = 0, 1, 2, \dots$ **do**
  $x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)})$ for $i = 1, 2, \dots, n$.
end for

- Write $x^{(k)} = [x_i^{(k)}]$ and $A = D - L - U$ where $D, -L, -U$ are the diagonal, strictly lower triangular, strictly upper triangular parts of A respectively. Show that $x^{(k+1)} = (D - L)^{-1}Ux^{(k)} + (D - L)^{-1}b$.
- Prove that if A is diagonally dominant, then A is invertible, and the Gauss-Seidel iteration converges, namely $\lim_{k\to\infty} x^{(k)} = A^{-1}b$ for any $x^{(0)}$. (A matrix A is called diagonally dominant, if $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$ for $i = 1, \dots, n$.)
- Prove that if A is symmetric and positive definite, then A is invertible, and the Gauss-Seidel iteration converges.

(3) (20 points) Given the Hilbert matrix $H_n = \left[\frac{1}{i+j-1}\right]_{i,j=1,\dots,n}$. Prove:
- H_n is positive definite.
- the spectral radius $\rho(H_n)$ of H_n is strictly monotonically increasing with respect to n.
- $\rho(H_n) \to \pi$ as $n \to \infty$.

(4) (10 points) Apply the following three-step method with a parameter $\theta \geqslant 0$ for solving the heat equation $u_t - a^2u_{xx} = 0$ (Cauchy or periodic problem):

$$(1+\theta)\frac{u_j^{n+1} - u_j^n}{k} - \theta\frac{u_j^n - u_j^{n-1}}{k} = a^2\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}.$$

Give its truncation error and stability. Particularly, give the value of θ such that the truncation error attains its highest order.

(5) (15 points) Denote the grid $I_h = \{x_j\}_{j=0}^M$, $x_0 < x_1 < \dots < x_M$. Set $u = \{u_j\}_{j=0}^M$ as a grid function on I_h. Suppose

$$Lu_j = -(a_ju_{j-1} - b_ju_j + c_ju_{j+1}) + q_ju_j, \quad j = 1, \dots, M-1,$$

where $a_j, b_j, c_j > 0$, $q_j \geqslant 0$ and $a_j + c_j \leqslant b_j$.

- Assume $Lu_j \leqslant 0$ for all $1 \leqslant j \leqslant M-1$. Show that u_j can't attain positive maximum at inner points $(1 \leqslant j \leqslant M-1)$ unless $u_j \equiv C$;
- Suppose $d_j = b_j - a_j - c_j + q_j > 0 (j = 1, \ldots, M-1)$. Show the solution of the difference equation

$$Lu_j = \varphi_j, \quad j = 1, \ldots, M-1; \qquad u_0 = u_M = 0,$$

satisfies $\|u\|_\infty = \max_j |u_j| \leqslant \max_j \frac{|\varphi_j|}{d_j}$.

(6) (10 points) Let $S \subset \mathbb{R}^n$ be a non-empty closed convex set, $f \in \mathcal{C}^2(S)$ is convex. For any $x^0 \in S$, define $L(f(x^0)) = \{x \in S | f(x) \leq f(x^0)\}$, if there exists some $m > 0$ such that

$$d^\top \nabla^2 f(x) d \geq m\|d\|^2, \quad \forall x \in L(f(x^0)), d \in \mathbb{R}^n.$$

Prove that $L(f(x^0))$ is a bounded and closed convex set.

(7) (10 points) Given a linear programming (LP) :

$$\begin{aligned} \max\ & 3x_1 + 2x_2 + 15x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 + 5x_3 \leq 18, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

- Write down the standard LP form.
- Compute the optimal solution and optimal value of the problem.