清华大学考试试题专用纸

考试课程 Subject: 求真书院代数博士生资格考 Algebra Qualifying Exam for Qiuzhen College

姓名 Name: _____

学号 Student ID: _____

- Exam Time: February 2026 (3 hours)
- This exam has 4 pages of 7 basic problems and 2 out of 6 advanced problems (100 points).
- Candidates are expected to adhere to the examination discipline by default, and those who do not comply will bear the consequences themselves.
- To receive full credit, all answers must provide necessary **details**, **proofs**, and **justifications**.
- You are strongly encouraged to write your answers **in English**.
- **第 8-13 题为提高题，你需要选择其中任意两题作答。请在答题纸首页清晰画出并填写下列表格，以注明需评分的两道题目。未在首页列出的题目将不予评分。**
  Q.8-13 are advanced problems and **selective**. Choose and answer **two** of them. No other answers will be graded. **Draw and fill the following table on the front page of the answer sheet to declare which two questions need to be graded.**

Advanced Problems		
Q.		
Score		

In the following problems, $\mathbf{Q}$ is the field of rational numbers, $\mathbf{R}$ is the field of real numbers, $\mathbf{C}$ is the field of complex numbers, $\mathbf{Z}$ is the ring of integers, $\mathbf{F}_q$ is the finite field consisting of q elements. **In all the problems, a ring means a commutative ring with multiplicative unit**.

1 基础题 Basic Questions

Q. 1 (10 points). *Let $n \geq k \geq 1$ be positive integers. For a $\mathbf{C}$-linear endomorphism $f\colon V \to V$ of an n-dimensional $\mathbf{C}$-vector space, prove that the following are equivalent.*

(a) f is an isomorphism;

(b) the induced endomorphism $\otimes^k f\colon \bigotimes^k V \to \bigotimes^k V$ on the k-th tensor power is an isomorphism;

(c) the induced endomorphism $\wedge^k f\colon \bigwedge^k V \to \bigwedge^k V$ on the k-th exterior power is an isomorphism.

Q. 2 (10 points). *Let* $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$.

1. *Find the eigenvalues of A.*
   Hint: one of the eigenvalues is an integer.

2. *Consider the set*
$$D = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{Q}^3 \;\middle|\; A^{100} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - A \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{Z}^3, \quad 0 \leq x, y, z < 1 \right\}.$$
   Find the cardinality of D.

Q. 3 (10 points).

1. *Show that the symmetric group S_4 has no normal Sylow subgroups.*

2. *Show that every group of order 24 with no normal Sylow subgroups must be isomorphic to S_4.*

Q. 4 (10 points). *Let $f(x) = x^5 - 11x - 1 \in \mathbf{Q}[x]$.*

1. *Show that $f(x)$ is irreducible over $\mathbf{Q}$.*

2. *Let G denote the Galois group of f. Show that G is isomorphic to the symmetric group S_5.*

Q. 5 (10 points). *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Assume that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some $n > 0$. Show that R is Artinian.*

Q. 6 (10 points). *Consider the $\mathbf{Z}[x, y]$-algebras*
$$A = \mathbf{Z}[x, y]/(x^3 + xy + y^2 + 7x - 4, 10x - 5y + 1) \quad \textit{and} \quad B = \mathbf{Z}[x, y]/(2x - y - 1).$$
Set $C = A \otimes_{\mathbf{Z}[x,y]} B$. Let $f \in C$ denote the image of x and write C_f for the localization $S^{-1}C$ of C with respect to $S = \{f^n \mid n \geq 0\}$. Find all the elements $e \in C_f$ with $e^2 = e$.

Q. 7 (10 points). *Let G be a finite group. Let $\{x_i\}_{i \in I}$ be a set of representatives for the conjugacy classes of G, i.e., it contains exactly one element in each conjugacy class. Let $\chi : G \to \mathbf{C}^\times$ be an irreducible character of G. Write $n = \sum_{i \in I} \chi(x_i)$. Show that n is an integer greater or equal to zero.*

2 高等题 Advanced Questions

Q. 8 (15 points). *Let t be a variable. Assume that there exist non-constant elements x, y in the field $\mathbf{C}(t)$ of rational functions such that*

$$y^n = \prod_{j=1}^{2026}(x-j)$$

for some $n \in \mathbf{Z}_{\geq 1}$. Show that $n = 1$.

Q. 9 (15 points). *Let K be an algebraically closed field and X the hypersurface in $\mathbf{P}^3_K$ defined by the homogeneous equation*

$$x^2w - y^2z = 0.$$

where (x, y, z, w) are homogeneous coordinates on $\mathbf{P}^3_K$. Prove that X is a rational variety, i.e., its function field is $K(t_1, t_2)$, where t_1, t_2 are algebraically independent over K.

Q. 10 (15 points). *Let $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.*

1. *Find the ring $\mathcal{O}_K$ of integers of K.*
2. *Regard the unit group $\mathcal{O}_K^\times$ as a $\mathbf{Z}$-module. Find a $\mathbf{Q}$-basis of $\mathcal{O}_K^\times \otimes_{\mathbf{Z}} \mathbf{Q}$.*

Q. 11 (15 points).

1. *Let K be a number field and let $\mathcal{O}_K$ denote its ring of integers. Show that there exists a nonzero $f \in \mathcal{O}_K$ such that $\mathcal{O}_K[1/f]$ is a PID.*
2. *Let $K = \mathbf{Q}(\sqrt{-83})$. Prove or disprove the following statement.*

   There exists a prime $p \in \mathbf{Z}$ such that $\mathcal{O}_K[1/p]$ is a PID.

Q. 12 (15 points). *Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{C})$. Let e_{ij} denote the elementary marix whose (i, j) entry is 1, and other entries are zero. Let $\mathfrak{b}$ be the Borel subalgebra consisting of upper-triangular matrices. Let $\mathfrak{t}$ be the Cartan subalgebra consisting of diagonal matrices. Let $V = \mathbf{C}^n$ be the standard representation of $\mathfrak{g}$. Write $\epsilon_1, ..., \epsilon_n$ for the weights of this representation. Recall that for each $\lambda \in \mathfrak{t}^*$, we have a Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$, where $\mathfrak{b}$ acts on $\mathbf{C}_\lambda$ through the quotient to $\mathfrak{t}$ and the latter acts by the weight λ.*

1. *Show that the module $M(\lambda) \otimes V$ admits a filtration whose successive quotients are Verma modules $M(\lambda + \epsilon_i)$ with i running over from 1 to n, each appearing exactly once.*
2. *Show that*

$$X : M(\lambda) \otimes V \to M(\lambda) \otimes V, \quad m \otimes v \mapsto \sum_{1 \leqslant i,j \leqslant n} e_{ij}(m) \otimes e_{ji}(v)$$

   is a $\mathfrak{g}$-module homomorphism.

   Hint: you might want to connect them to Casimir elements.)

3. *Compute the generalised eigenvalues of X acting on $M(\lambda) \otimes V$. Show that each generalised eigenspace is again a $\mathfrak{g}$-representation filtered by Verma modules. Describe the successive quotients in this filtration for each possible eigenvalue.*

Q. 13 (15 points)**.** *Let (W, S) be a dihedral Coxeter group. That is*

$$W = \langle s, t \mid s^2 = t^2 = (st)^{m_{st}} = e \rangle$$

where $e \in W$ is the identity, and $m_{st} \in \{2, 3, 4, ..., \infty\}$.

1. *Let $\mathcal{H}$ be the Hecke algebra associated with the group (W, S). Write down the definition of $\mathcal{H}$ as a $\mathbf{Z}[v, v^{-1}]$-algebra in terms of generators and relations.*

2. *Write down the definition of the Kazhdan-Lusztig basis of $\mathcal{H}$. Denote elements in this basis by b_x for $x \in W$.*

3. *Given $0 \leqslant m \leqslant m_{st}$ write $st(m)$ for the element $stst...$ where m terms appear, and similar for $ts(m)$. For example $st(0) = e$, $ts(1) = t$, $st(2) = st$, $ts(3) = tst$, etc. Give explicit description of all elements in W and describe the Bruhat order on W explicitly.*

4. *Compute the expansion of b_x in terms of standard basis in $\mathcal{H}$ for all $x \in W$ and justify your answer.*