

Qualifying Exam: 2026 Fall

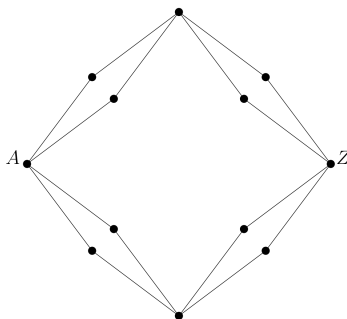
考试课程: Probability & Statistics 姓名: _____ 学号: _____

- There are 11 problems in this exam (5 pages). You need to choose 8 of them to solve. If you select more than 8, only the first 8 that you have worked on will be graded. Note that 4 of the problems are worth 15 points each and the rest 10 points each.
- You must follow all the rules of exam taking. Misconducts will be subject to proper disciplinary actions by the Center.
- You must provide all necessary details for full credits. A final answer with no or little explanation/derivation, even if correct, receives a minimal credit.
- $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of positive integers. $\xrightarrow{(d)}$ and $\stackrel{(d)}{=}$ mean “converges in distribution” and “equal in distribution”, respectively.
- For $\mu \in \mathbb{R}, \sigma > 0$, the 1-dimensional normal distribution $\mathcal{N}(\mu, \sigma^2)$ is defined by the density function

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

- Facts about Poisson(λ), $\lambda > 0$:
pmf: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, \dots$
 $EX = \lambda, \text{Var } X = \lambda$.
mgf: $M_X(s) = \exp\{\lambda(e^s - 1)\}, s \in \mathbb{R}$.
- Facts about Exp(θ), $\theta > 0$:
pdf: $f(x) = \theta^{-1}e^{-x/\theta}, x > 0$.
 $EX = \theta, \text{Var } X = \theta^2$.
mgf: $M_X(s) = (1 - s\theta)^{-1}, s < 1/\theta$.
- Facts about Gamma(α, β), $\alpha > 0, \beta > 0$:
pdf: $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}, x > 0$.
 $EX = \alpha\beta, \text{Var } X = \alpha\beta^2$.
mgf: $M_X(s) = (1 - s\beta)^{-\alpha}, s < 1/\beta$.

1. (10 points) For the graph in Figure 1, every edge independently has probability $p \in (0, 1)$ to remain, and probability $(1-p)$ to be deleted. What is the probability to have a path connecting A and Z in the graph?



2. (10 points) Let X and N be two independent random variables. Suppose X follow the standard normal distribution $\mathcal{N}(0, 1)$, and N follows the Poisson distribution of parameter $\lambda > 0$.
- (a) Find the value λ to minimize $\mathbb{E}[X^N]$.
- (b) Is there any $\lambda > 0$ such that $\text{Var}[X^N] < \infty$? Justify your answer and write down the expression of $\text{Var}[X^N]$ if it exists.
3. (10 points) Let $(B_t)_{t \geq 0}$ be a standard Brownian motion, let $\mu > 0$, and define

$$X_t = B_t - \mu t, \quad S = \sup_{t \geq 0} X_t.$$

- (a) Prove that $X_t \rightarrow -\infty$ almost surely as $t \rightarrow \infty$ and that

$$Z_t = \exp(2\mu X_t)$$

is a martingale.

- (b) Prove that, for every $a \geq 0$,

$$\mathbb{P}(S \geq a) = e^{-2\mu a}.$$

4. (10 points) In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there are independent random variables $(X_n)_{n \in \mathbb{N}_+}$ which follow the Poisson distribution $X_n \sim \text{Poisson}(\frac{1}{n^2})$. Denote by $S_n := \sum_{k=1}^n X_k$.
- (a) Prove that S_n converges in distribution to a limit Y . Identify the distribution of Y .
- (b) Prove that there exists a random variable Y' living in the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that S_n converges almost surely and in L^1 to Y' .
- (c) Show that Y' has the same distribution as Y .

5. (15 points) Let $(X_n)_{n \geq 0}$ be an irreducible Markov chain on a finite state space E , with transition matrix P and stationary distribution π . Fix $j \in E$, and denote by $\mathbb{P}_j, \mathbb{E}_j$ respectively the probability and expectation for the chain starting from $X_0 = j$. Let

$$T_j^+ := \inf\{n \geq 1 : X_n = j\}.$$

For $i \in E$, define

$$\nu_i := \mathbb{E}_j \left[\sum_{n=0}^{T_j^+ - 1} \mathbf{1}_{\{X_n=i\}} \right].$$

- (a) Prove that $\mathbb{E}_j[T_j^+] < \infty$.
- (b) Prove that the row vector $\nu = (\nu_i)_{i \in E}$ satisfies $\nu P = \nu$.
- (c) Deduce that

$$\nu_i = \frac{\pi_i}{\pi_j} \quad \text{for every } i \in E, \quad \mathbb{E}_j[T_j^+] = \frac{1}{\pi_j}.$$

6. (15 points) Let $(B_t)_{0 \leq t \leq 1}$ be standard Brownian motion and define

$$W_t = B_t - tB_1, \quad 0 \leq t \leq 1.$$

- (a) Prove that W is independent of B_1 and that

$$\mathbb{E}[W_s W_t] = \min(s, t) - st.$$

Thus W is a standard Brownian bridge.

- (b) If $M = \max_{0 \leq t \leq 1} W_t$, prove that, for $x \geq 0$,

$$\mathbb{P}(M < x) = 1 - e^{-2x^2}.$$

7. (10 points) Let $X_1, \dots, X_n$ be independent with density

$$f_\theta(x) = \mathbf{1}_{[\theta, \theta+1]}(x), \quad \theta \in \mathbb{R},$$

where θ is unknown. Write

$$X_{(1)} = \min_i X_i, \quad X_{(n)} = \max_i X_i.$$

- (a) Determine the set of the maximum-likelihood estimates of θ . Define $\hat{\theta}_n$ to be the midpoint of this set.
 - (b) Find the limiting distribution of $n(\hat{\theta}_n - \theta)$, including its density.
8. (10 points) Let $X_1, \dots, X_n$ be i.i.d. $\text{Poisson}(\lambda)$, $\lambda > 0$, and set $T = \sum_{i=1}^n X_i$.

- (a) Let $g(\lambda) = P_\lambda(X_1 = 1) = \lambda e^{-\lambda}$. Derive the UMVUE of $g(\lambda)$, and denote it by $\hat{g}_n$.
- (b) Now suppose that zeros cannot be observed: $Y_1, \dots, Y_n$ are i.i.d. from the zero-truncated Poisson distribution

$$P(Y = k) = \frac{e^{-\lambda} \lambda^k}{k! (1 - e^{-\lambda})}, \quad k = 1, 2, \dots$$

Let N_1 and N_2 denote the numbers of Y_i equal to 1 and 2, respectively, and define $\tilde{\lambda}_n = 2N_2/N_1$ (define $\tilde{\lambda}_n = 0$ if $N_1 = 0$). Show that $\tilde{\lambda}_n$ is a consistent estimator of λ , and find the limiting distribution of $\sqrt{n}(\tilde{\lambda}_n - \lambda)$.

9. (15 points) Let $X_1, \dots, X_n$ be i.i.d. $\text{Exp}(\theta)$, $\theta > 0$, and let $T = \sum_{i=1}^n X_i$. Consider estimating θ under the loss

$$L(\theta, d) = \left(\frac{d}{\theta} - 1\right)^2.$$

- (a) Show that every estimator cT ($c > 0$) has constant risk, find the value c^* minimising the risk, and compute the minimal risk.
- (b) Put the prior $\lambda = 1/\theta \sim \text{Gamma}(\alpha, \beta)$. Find the Bayes estimator $\delta_{\alpha, \beta}$ under the loss L .
- (c) Compute the Bayes risk of $\delta_{\alpha, \beta}$ and show that it does not depend on β .
- (d) Is c^*T minimax under L ? Is the UMVUE T/n admissible under L ? Please justify your answer.

10. (15 points) Let

$$X_1, \dots, X_n \stackrel{\text{iid}}{\sim} P_{\pi, \lambda},$$

where

$$P_{\pi, \lambda}(X = 0) = \pi,$$

and, given $X \neq 0$, it has Lebesgue density on $x > 0$,

$$f_{\lambda}(x) = \lambda e^{-\lambda x},$$

with

$$0 \leq \pi \leq 1, \quad \lambda > 0.$$

Let

$$N_0 = \sum_{i=1}^n \mathbf{1}\{X_i = 0\}, \quad S = \sum_{i=1}^n X_i.$$

denote some summary statistics.

- (a) Carefully express the likelihood function $L(\pi, \lambda)$. Specify the dominating measure you have chosen.
 - (b) Find the MLE of (π, λ) , say $(\hat{\pi}, \hat{\lambda})$.
 - (c) Identify the limiting distribution of $(N_0/n, S/n)$.
 - (d) Derive the asymptotic distribution of the MLE $(\hat{\pi}, \hat{\lambda})$ based on the last step. What do you learn from the expression of the limiting covariance matrix?
11. (10 points) Let $X_1, \dots, X_n$ be i.i.d. observations with population density p on the real line. Suppose that it is known that p is either p_1 or p_2 , with $p_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta_1)^2}{2}\right\}$ and $p_2(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta_2)^2}{2}\right\}$ for known constants $-\infty < \theta_1 < \theta_2 < \infty$. We consider squared L_2 loss for the estimation of p . Given an estimator $\hat{p}$ based on the sample, the larger risk

$$R(\hat{p}, n) = \max \left(E_{p_1} \int (p_1(x) - \hat{p}(x))^2 dx, E_{p_2} \int (p_2(x) - \hat{p}(x))^2 dx \right)$$

is considered to quantify the performance of $\hat{p}$.

One natural approach is to do estimation of p based on selection between the two densities. Note that a selection rule between p_1 and p_2 corresponds to a testing rule of the hypothesis $H_0 : p = p_1$ versus $H_1 : p = p_2$. Let ϕ denote a testing rule and let C_ϕ and A_ϕ be its rejection and acceptance regions, respectively. Then the estimator based on testing (selection) is

$$p_\phi(x) = p_1(x)\mathbf{1}_{A_\phi} + p_2(x)\mathbf{1}_{C_\phi}.$$

(a) Show

$$R(p_\phi, n) = \max(P_{p_1}(C_\phi), P_{p_2}(A_\phi)) \int (p_1(x) - p_2(x))^2 dx.$$

(b) Find the best test ϕ^* that minimizes $R(p_\phi, n)$ and prove your result. Simplify your answer as much as possible.