清华大学考试试题专用纸

姓名: ____________ 学号: ____________

Everyone needs to solve Problem 1, 2, 3, 4 (10 points each). Please select four problems in the following table and solve them (15 points each).

Problem 5	Problem 6	Problem 7	Problem 8	Problem 9	Problem 10

1. Construct two covering maps, $p : X \to Y$ and $q : Y \to Z$, among topological spaces, such that $q \circ p : X \to Z$ is not a covering map. Justify your answer.

2. Consider $\mathbb{RP}^n$, $n \geq 3$, with the standard metric induced from the standard sphere S^n of sectional curvature 1 via the antipodal quotient. Fix a point $[p] \in \mathbb{RP}^n$, where $p \in S^n$. Describe the cut locus $\mathrm{Cut}([p]) \subset \mathbb{RP}^n$ explicitly.

3. For each natural number $k \geq 0$, construct a closed connected 3-manifold such that

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \mathbb{Z}/2.$$

4. Let (M^3, g) be a Riemannian 3-manifold and fix $p \in M$. Choose an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM that diagonalizes $\mathrm{Ric}_p : T_pM \to T_pM$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

   (a) Show that $\sec_p > 0$ if and only if $\lambda_i + \lambda_j > \lambda_k$ for all distinct i, j, k. In particular, $\mathrm{Ric}_p > 0$ (positive definite) does not imply $\sec_p > 0$.

   (b) Does $\mathrm{Ric}_p = 0$ imply $\sec_p = 0$ in dimension 3? Justify your answer.

5. Let $S \subset \mathbb{R}^3$ be a smoothly embedded closed surface. Prove that S is orientable.

6. Let (M, g) be a closed (compact, no boundary) Riemannian manifold and let X be a Killing vector field. Show that if $\mathrm{Ric} < 0$ at some point of M, then $X \equiv 0$.

7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex polynomial in one variable. Let $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the unique smooth map that is compatible with f. (Namely, let $i : \mathbb{C} \to \mathbb{CP}^1$ be the natural inclusion, then $F \circ i = i \circ f$.)

   Prove that the mapping degree of F equals the degree of f as a polynomial.

8. Let $\Omega = \mathbb{R}^n \setminus \{0\}$ and $u(r) = 1 + r^{2-n}$ (with $r = |x|$), $n \geq 3$. Consider the metric $g_u = u^{\frac{4}{n-2}}\, \delta$, where δ is the Euclidean metric. Decide whether (Ω, g_u) is complete, and justify your answer.

9. Let M be a closed oriented manifold of dimension n. Let $S \subset \mathrm{int}(M)$ be a compact embedded submanifold of dimension k. Prove that

$$H^{n-i}(S, \mathbb{Z}) \cong H_i(M, M \setminus S, \mathbb{Z}) \quad \text{for all } i.$$

10. Let (Σ, g) be a Riemannian 2-dimensional surface and fix $p \in \Sigma$. For small $r > 0$, let $S_r(p) = \{q \in \Sigma \mid d(p, q) = r\}$ be the circle of radius r around p. Assume that there exists $r_0 > 0$ such that $\text{Length}(S_r(p)) \leq 2\pi r$ for all $r \in (0, r_0)$. Prove that the Gauss curvature at p satisfies $K_p \geq 0$.