Name:_________________ Student ID: _________________

Qualifying Exam-Analysis

Spring 2025

1. (10 pts) Compute the Fourier transform $\hat{u}(\xi)$ of $u(x) = \dfrac{x}{1+x^2}$. (Here we use the following definition for Fourier transform of L^1 functions

$$\hat{u}(\xi) = \int_{\mathbb{R}} e^{-ix\cdot\xi} u(x) dx$$

and extend to tempered distributions.)

2. (10 pts) Compute the Fourier series

$$\sum_{n\in\mathbb{Z}} a_n e^{inx}$$

of the characteristic function f of $[-1, 1]$, that is,

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } 1 < |x| \le \pi \end{cases},$$

and find all points $x \in [-\pi, \pi]$ for which this series converges absolutely.

3. (10 pts) For all $u_0 \in C_c^\infty(\mathbb{R})$, we define $u(t, x) \in C^\infty(\mathbb{R}^2)$ as follows

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(t\xi^3 + x\xi)} \widehat{u_0}(\xi)\, \mathrm{d}\xi.$$

Show that for all $x_0 \in \mathbb{R}$, the function $t \mapsto \partial_x u(t, x_0)$ belongs to $L^2(\mathbb{R})$ and there exists a constant $c_0 > 0$ independent of x_0 and u, such that

$$\int_{\mathbb{R}} |\partial_x u(t, x_0)|^2 \, \mathrm{d}t = c_0 \int |u_0(x)|^2 \, \mathrm{d}x.$$

4. (10 pts) Let $f : \Omega \to \mathbb{C}$ be non-constant and holomorphic, where $\Omega \subset \mathbb{C}$ is an open set containing the closed unit disk $|z| \le 1$. Assume that $|f(w)| = 1$ whenever $|w| = 1$, show that $f(\Omega)$ contains the open unit disk. (Hint: reduce to the statement that $f(z) = 0$ has at least one root in the disk.)

5. (15 pts) Consider the following second order linear equation for $u = u(x)$:

$$x\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = 0.$$

(i) Prove that all nontrivial real-valued solutions have infinite number of zeroes on $(1, \infty)$.

(ii) Is it true that all nontrivial real-valued solutions must have finite number of zeroes on $(0, 1)$? Prove your answer.

6. (15 pts) Let $p \in [1, \infty)$ and $\{f_n\}_{n=1}^{\infty}$ a sequence of functions in $L^p(\mathbb{R})$ such that $f_n \to f$ a.e. and $f \in L^p(\mathbb{R})$.

(i) If $p \in (1, \infty)$, prove that if $\sup_n \|f_n\|_{L^p} < \infty$, then f_n converges to f weakly, i.e. for any $g \in L^q(\mathbb{R})$ with $q = \frac{p}{p-1}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n g dx = \int_{\mathbb{R}} f g dx.$$

(ii) Show that the result in (1) does not hold when $p = 1$, $q = \infty$.

7. (15 pts) Assume that $n \geq 2$, $p \in (1, 2)$, and $q \in [p, +\infty]$.

(i) Show that there exists a constant C (may depend on p, q, n) such that for all radial function $f \in C_c^{\infty}(\mathbb{R}^n \backslash \overline{B})$, there holds:

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}.$$

Here $\overline{B}$ is the closed unit ball in $\mathbb{R}^n$

(ii) Show that F has a compact closure in X. Here X is the closure of $\{f \in C_c^{\infty}(\mathbb{R}^n \backslash \overline{B}) : f \text{ is radial}\}$ under $L^{\infty}(\mathbb{R}^n)$-norm and

$$F = \left\{f \in C_c^{\infty}(\mathbb{R}^n \backslash \overline{B}) : f \text{ is radial and } \|f\|_{W^{1,p}(\mathbb{R}^n)} \leq 1\right\}$$

8. (15 pts) Given a domain $\Omega \subset \mathbb{C}$ and a point $p \in \Omega$, define

$$c(p) = \sup\{|f'(p)| : \ f \in (\Omega, D)_p\},$$

where D is the open unit disk $|z| < 1$ and $(\Omega, D)_p$ is the set of holomorphic maps $f : \Omega \to D$ with $f(p) = 0$.

(i) Find the value of $c(p)$ when $\Omega = \mathbb{C} \setminus \{0, 1, 2\}$.

(ii) Find the value of $c(p)$ when $\Omega = D$.