清华大学考试试题专用纸

姓名: ___________ 学号: ___________

Everyone needs to solve Problem 1, 2, 3, 4 (10 points each). Please select four problems in the following table and solve them (15 points each).

Problem 5	Problem 6	Problem 7	Problem 8	Problem 9	Problem 10

1. Consider the topological space X given by the gluing of the following polygonal pieces:

$$14\underline{6}, \quad 52\underline{1}53, \quad \underline{234}6.$$

Here the word $14\underline{6}$ means the labeling scheme for the 3-gon is $e_1e_4e_6^{-1}$, and so for the other words. Each $e_i, 1 \le i \le 6$ is oriented counter-clock-wisely and the underline means the orientation of the edge is clock-wise.

Compute the homotopy groups $\pi_i(X)$ and singular homology groups $H_i(X;\mathbb{Z})$ for all $i \ge 1$.

2. Let M be a Riemannian manifold and $\mathcal{K}$ be a set of isometries of M. Suppose that F is a nonempty subset of M which is fixed by every element in $\mathcal{K}$. Show that each connected component of F is a closed totally geodesic submanifold of M.

3. Let X be a compact Lie group of dimension ≥ 1. Show that the Euler characteristics of X vanishes.

4. Let $\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$ be the upper half-space with the metric defined by $ds^2 = \frac{1}{x_n^2}(dx_1^2 + \ldots + dx_n^2)$. Let Σ be a k-dimensional minimal submanifold of $\mathbb{H}^n$, where $k < n$. Show that $\mathrm{Ric}(\Sigma) \le 0$, i.e., $\forall$ vector field X on Σ, we have $\mathrm{Ric}(X, X) \le 0$.

5. Suppose X is a contractible, compact manifold of dimension $n \ge 1$. Prove that $H_i(\partial X;\mathbb{Z}) = H_i(S^{n-1};\mathbb{Z})$ for all i.

6. Let (M, g) be a Riemannian manifold and $\pi : \tilde{M} \to M$ be a covering map with the pullback metric π^*g on $\tilde{M}$.

   (a) Show that M is complete if and only if $\tilde{M}$ is complete.

   (b) Is the conclusion in (a) true if π is only assumed to be a local isometry? If so, provide a proof. If not, provide a counterexample.

7. Let X be a connected finite-dimensional CW complex, $\tilde{X}$ its universal cover. Let $K(\pi_1(X), 1)$ be the Eilenberg-McLane space (i.e., the space whose fundamental group is $\pi_1(X)$ and whose higher homotopy groups vanish.) Let $Y = \tilde{X} \times K(\pi_1(X), 1)$. Show that:

   (a) X and Y have isomorphic homotopy groups;

(b) If $\pi_1(X)$ contains an element of order 2, then X and Y are not homotopy equivalent.

8. Let M be a complete Riemannian manifold with nonnegative sectional curvature and $\gamma : [0,1] \to M$ be a minimizing geodesic with $\gamma(0) = p, \gamma(1) = q$. Suppose that X is a parallel vector field along γ such that $X \perp \gamma'$.

   (a) Show that for sufficiently small $t > 0$, $\mathrm{dist}(\exp_p(tX(0)), \exp_q(tX(1))) \leq \mathrm{dist}(p, q)$.

   (b) State the condition for which the equality in (a) holds (no proof needed).

9. Let X be the space of "a line contained in a plane in $\mathbb{C}^3$", namely a point $F \in X$ is described as a sequence

$$\{0\} \subset l \subset P \subset \mathbb{C}^3,$$

   where P is a complex plane and l is a complex line.

   (a) The $\mathrm{GL}(3, \mathbb{C})$-action on $\mathbb{C}^3$ induces an action on X. Describe the stabilizer of the action and prove that X is a smooth manifold.

   (b) Compute the cohomology ring $H^*(X; \mathbb{R})$.

10. Let (M, g) be a complete compact oriented Riemannian manifold. We say a vector field X on M is a conformal Killing field if $\mathcal{L}_X g = fg$ for some smooth function f, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. If $\mathrm{Ric}(M) \leq 0$, show that every conformal Killing field on M is parallel. (Hint: Consider $\nabla_X X - \mathrm{div}(X)X$.)