Name:________________ Student ID:________________

Qualifying Exam-Analysis

Fall 2025

1. (10 pts) Show that the following limit u exists in the sense of distribution (in $\mathscr{D}'(\mathbb{R}^2)$) and compute its Fourier transform $\hat{u}(\xi)$:

$$u = \lim_{\varepsilon \to 0+} (x_1^2 + \varepsilon + ix_2)^{-2}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

(Here we use the following definition for Fourier transform of L^1 functions

$$\hat{u}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} u(x)dx, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$$

and extend to tempered distributions.)

2. (10 pts) Assume $0 \in \Omega$ is an open connected subset of the unit disk $\mathbb{D}$. Let $\mathcal{F}$ be the family of all the holomorphic maps $f : \Omega \to \mathbb{D}$ such that f is injective, $f(0) = 0$ and $f(\Omega) \subset \mathbb{D}$. Prove that there is a holomorphic map $g \in \mathcal{F}$ such that

$$|g'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|.$$

3. (10 pts) Find the C^1 solution $u = u(x, y)$ to the following equation and the maximum domain of the solution determined by the initial condition with the method of characteristics

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) = u, \quad u(0, y) = y.$$

4. (10 pts) Given two real numbers a, b, consider the problem on finding all the continuous real-valued functions u in the closed disc $\overline{\mathbb{D}} = \{|z| \leq 1\}$, such that u is harmonic in $\mathbb{D} \setminus \{0\} = \{0 < |z| < 1\}$, $u(0) = a$, and $u(z) = b$ when $|z| = 1$. Discuss the solvability of the problem, and if it is solvable, find all the solutions. Prove your answer. (Hint: assume the solution exists, you may first consider the symmetry of the solution.)

5. (15 pts) Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of Lebesgue measurable function on $\mathbb{R}$ such that

- $f_n \to f$ almost everywhere, where $f : \mathbb{R} \to \mathbb{R}$;
- there exists a Lebesgue integrable function $g : \mathbb{R} \to [0, \infty)$ such that for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$.

(i) Show that for any $\varepsilon > 0$, there exists a subset $E \subset \mathbb{R}$ with Lebesgue measure $m(E) < \varepsilon$ such that f_n converges uniformly to f on $\mathbb{R} \setminus E$.

(ii) Is it always possible to find $E \subset \mathbb{R}$ with Lebesgue measure $m(E) = 0$ such that f_n converges uniformly to f on $\mathbb{R} \setminus E$? Prove if true and give a counterexample otherwise.

6. (15 pts) Let $\omega > 0$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with period 2π. Consider the equation

$$\frac{d^2u}{dx^2} + \omega^2 u = g(x).$$

(i) Show that if ω is not an integer, then there is a unique solution with period 2π and all solutions are bounded.

(ii) When ω is a positive integer, find and prove the sufficient and necessary condition for g such that there is a solution with period 2π.

7. (15 pts) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.

(i) Show that the embedding $C^1([0,1], H^1(\mathbb{R}^n)) \hookrightarrow C([0,1], L^2(\Omega))$ is compact.

(ii) Show that the embedding $C([0,1], H^1(\mathbb{R}^n)) \hookrightarrow C([0,1], L^2(\Omega))$ is not compact by providing a counter example.

Here for a Banach space X, we denote by $C^k([0,1], X)$ the space consisting of all C^k-maps f from $[0,1]$ to X. The norm of $C^k([0,1], X)$ is given by

$$\|f\|_{C^k([0,1],X)} := \sup_{t \in [0,1], 0 \leq \ell \leq k} \|\partial_t^\ell f(t)\|_X.$$

8. (15 pts) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Consider the following two minimizing problems:

$$\mathcal{I} = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_{L^2(\Omega)}^2 \|u\|_{L^2(\Omega)}^2}{\|u\|_{L^4(\Omega)}^4},$$

$$\mathcal{J}_M = \inf_{\substack{u \in H_0^1(\Omega), \\ \|u\|_{L^2(\Omega)} = M}} \left(\frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{4} \int_\Omega |u|^4 \right), \quad \forall M > 0.$$

(i) Show that the the first minimizing problem has a minimizer in $H_0^1(\Omega)$.

(ii) Show that there exists $0 < M_0 < +\infty$, such that for all $M > M_0$, we have $\mathcal{J}_M = -\infty$, and for all $0 < M \le M_0$, we have $\mathcal{J}_M \ge 0$.