清华大学考试试题专用纸

考试课程 Subject: 求真书院代数博士生资格考 Algebra Qualifying Exam for Qiuzhen College

姓名 Name: ____________________

学号 Student ID: ____________________

- Exam Time: September, 2025 (3 hours)
- This exam has 3 pages of 7 basic problems and 2 out of 6 advanced problems (100 points).
- Candidates are expected to adhere to the examination discipline by default, and those who do not comply will bear the consequences themselves.
- All answers must include necessary details, proofs, and justifications.
- *You are strongly encouraged to write your answers in English.*
- **第 8-13 题为提高题，你需要选择其中任意两题作答。请在答题纸首页清晰画出并填写下列表格，以注明需评分的两道题目。未在首页列出的题目将不予评分。**
  Q. 8-13 are advanced problems and **selective**. Choose and answer **two** of them. No other answers will be graded. **Draw and fill the following table on the front page of the answer sheet to declare which two questions need to be graded.**

Advanced Problems		
Q.		
Score		

In the following problems, $\mathbf{Q}$ is the field of rational numbers, $\mathbf{R}$ is the field of real numbers, $\mathbf{C}$ is the field of complex numbers, $\mathbf{Z}$ is the ring of integers, $\mathbf{F}_q$ is the finite field consisting of q elements. **In all the problems, a ring means a commutative ring with multiplicative unit**.

1 基础题 Basic Questions

Q. 1 (10 points)**.** *For $n \geq 1$, let A, B be two $n \times n$ matrices over an algebraically closed field K such that $AB = BA$. Prove that A and B have a nonzero common eigenvector.*

Q. 2 (10 points)**.** *Let $R = K[x, y, z, w]$ denote the polynomial ring in four indeterminates over a field K. Define the ideal $I \subset R$ by*

$$I = \{f(x, y, z, w) \in R \mid f(s^3, s^2t, st^2, t^3) = 0 \quad \text{in the polynomial ring } K[s, t]\}.$$

Show that I cannot be generated by two elements as an ideal of R.

Q. 3 (10 points). *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of degree 4 and assume that f has exactly two real roots. Write G for the Galois group of the splitting field K of f over $\mathbf{Q}$.*

1. *Show that 8 divides the order $|G|$.*

2. *Show that G is not abelian.*

Q. 4 (10 points). *Let R be a ring and let $s \in R$. Set $S = \{1, s, s^2, ...\}$. Show that the following statements are equivalent:*

1. *The canonical ring morphism $R \to S^{-1}R$ is surjective.*

2. *There exists $m > 0$ such that $s^m R = s^M R$ for all $M \geqslant m$.*

3. *There exists $n > 0$ such that the ideal $s^n R$ is generated by an element e with $e^2 = e$.*

Q. 5 (10 points). *Let $A = \mathbf{C}[x, y]$ and $B = \mathbf{C}[u, v]$ be polynomial rings over $\mathbf{C}$, and define a $\mathbf{C}$-algebra homomorphism $f \colon A \to B$ by $f(x) = u$ and $f(y) = uv$. Prove or disprove each of the following statements.*

1. *B is a flat A-module.*

2. *B is an injective A-module.*

Q. 6 (10 points).

1. *Let A be a finite abelian group and χ a complex character of A. Show that*

$$\sum_{a \in A} |\chi(a)|^2 \geqslant |A| \cdot \chi(1).$$

2. *Let G be a finite group and A an abelian subgroup of G of index n. Let ψ be a complex irreducible character of G. Show that*

$$\psi(1) \leqslant n.$$

Q. 7 (10 points). *Let SU_2 be the Lie group of 2×2 special unitary matrices and $\mathfrak{su}_2$ the Lie algebra of SU_2. Prove that the image of the adjoint map $\mathrm{Ad} \colon \mathrm{SU}_2 \to \mathrm{GL}(\mathfrak{su}_2)$ is a Lie group isomorphic to $\mathrm{SO}(3, \mathbf{R})$.*

2 高等题 Advanced Questions

Q. 8 (15 points). *Suppose $r \leq n$ are positive integers. Let X be the subset of $M(n \times n, \mathbf{C})$ consisting of complex $n \times n$ matrices with rank at most r. Prove that X is Zariski closed. Find the number of irreducible components of X, and calculate the Krull dimension of each irreducible component of X.*

Q. 9 (15 points). *Let C be a genus one complex smooth projective curve. Take any point $p \in C$.*

1. *Prove that there exist rational functions f, g on C such that f has pole of order 2 at p, and g has pole of order 3 at p, and both f, g are regular on $C - \{p\}$.*

2. *Prove that C is isomorphic to a cubic curve in the complex projective plane* $\mathbf{CP}^2$.

Q. 10 (15 points). *Let* $K = \mathbf{Q}(\sqrt{82})$.

1. *Find the absolute value* $|\mathrm{disc}(K)|$ *of the discriminant* $\mathrm{disc}(K)$ *of* K.

2. *Determine the class group* $\mathrm{Cl}(K)$ *of* K. *Give a presentation of* $\mathrm{Cl}(K)$ *(i.e., a set of generators and relations).*

Q. 11 (15 points). *Let* $\mathbf{Q}_p$ *denote the field of p-adic numbers for a prime p and fix an algebraic closure* $\overline{\mathbf{Q}}_p$.

1. *Take* $\alpha \in \overline{\mathbf{Q}}_p$ *satisfying* $\alpha^{p-1} = -p$ *and set* $K \coloneqq \mathbf{Q}_p(\alpha) \subset \overline{\mathbf{Q}}_p$. *Prove that K is Galois over* $\mathbf{Q}_p$.

2. *Take* $\zeta \in \overline{\mathbf{Q}}_p$ *satisfying* $\zeta^p = 1$ *and* $\zeta \neq 1$, *and set* $L \coloneqq \mathbf{Q}_p(\zeta) \subset \overline{\mathbf{Q}}_p$. *Prove* $L = K$.

Q. 12 (15 points). *Consider the category* $\mathcal{O}$ *for* $\mathfrak{sl}_2$. *We identify the weight lattice of* $\mathfrak{sl}_2$ *with* $\mathbf{Z}$ *by sending the first fundamental weight to* 1. *We denote by* $L(\mu)$ *the simple* $\mathfrak{sl}_2$*-module of highest weight* μ. *In particular, the trivial representation* $\mathbf{C} = L(0)$. *Let* $M(\mu)$ *be the Verma module of highest weight* μ. *Let* $P(\mu)$ *be the projective cover of* $L(\mu)$.

1. *Show that there is only one* $\lambda \neq 0$ *such that* $L(\lambda)$ *has the same central character as* $L(0)$. *Determine the value of* λ.

2. *Write down composition series for the Verma modules* $M(0)$ *and* $M(\lambda)$ *for* λ *above and justify your answer.*

3. *Write down composition series for the projective modules* $P(0)$ *and* $P(\lambda)$ *for* λ *above and justify your answer.*

Q. 13 (15 points). *Consider the flag variety for* GL_3*:*

$$\mathcal{B} = \{V_\bullet = (0 = V_0 \subset V_1 \subset V_2 \subset V_3 = \mathbf{C}^3) \mid \dim(V_i) = i, \quad 1 \leqslant i \leqslant 3\}.$$

Let

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{B}_e$ *be the closed subvariety in* $\mathcal{B}$ *consisting of flags stablized by e, that is,*

$$\mathcal{B}_e = \{V_\bullet \in \mathcal{B} \mid \ e(V_i) \subset V_i, \quad 1 \leqslant i \leqslant 3\}.$$

1. *Compute the irreducible components of* $\mathcal{B}_e$.

2. *Compute the singular cohomology of* $\mathcal{B}_e$ *with coefficient in* $\mathbf{Q}$ *as a* $\mathbf{Q}$*-vector space.*