QUALIFY EXAM FOR APPLIED MATHEMATICS

1. (*15 points*) Assume that f is a periodic function with period 2π. Considering the integration $I(f) = \int_0^{2\pi} f(x)dx$, prove the following statements on the order of convergence of composite trapezoidal rule: $I_h^{Tr}(f) = h\sum_{i=1}^{n} f(i \cdot h)$, where $h = 2\pi/n$.
(i) Assume further that $f \in C^m([0, 2\pi])$, prove that the above rule achieves $m-$th order convergence, i.e. $\exists C > 0$ s.t. $\left|I(f) - I_h^{Tr}(f)\right| \leq C \cdot h^m$.
(ii) Assume further that f is analytic, prove that the above rule achieves exponential convergence.

2. (*15 points*) Let f be a continuous function that is $C^m(m \geq 1)$, such that $f(\alpha) = f^{(1)}(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)} \neq 0$. Answer the following questions:
(i) If $m = 1$, i.e. α is a simple zero of the function f, prove that there exists a neighborhood of α: $D(\alpha, \delta) = \{x : |x - \alpha| \leq \delta\}$, s.t. $\forall x_0 \in D(\alpha, \delta)$, the sequence generated by Newton's iteration converges to α, and the rate of convergence is quadratic.
(ii) If $m > 1$, i.e. α is a zero with multiplicity greater than 1, what is the rate of convergence of Newton's iteration? How about the modified Newton's iteration: $x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}$? Please prove your results.
(iii) If the multiplicity m is greater than 1 in general but the value is unknown, can you propose an iteration method that achieves superlinear convergence? (You don't need to prove the result.)

3. (*20 points*) The Sherman-Morrison formula (or the generalized version named Sherman-Morrison-Woodbury formula) is an useful tool in numerical linear algebra. It states: if $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, $u, v \in \mathbb{R}^n$ are two vectors satisfying $1 + v^T A^{-1}u \neq 0$, then the matrix $A + uv^T$ is invertible and can be computed by
$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u} .$$
(i) Prove the above statement.
(ii) If $\{s_j\}_{j=1}^n$ and $\{t_j\}_{j=1}^n$ are two sets of points in $[0, 1]$, $A \in \mathbb{R}^{n\times n}$ is a matrix defined by:
$$A_{j,k} = 4n * \delta_{j,k} + cos(t_j - s_k) ,$$
where $\delta_{j,k}$ is the Kronecker delta ($\delta_{j,k} = 1$ when $j = k$, $\delta_{j,k} = 0$ otherwise). $b \in \mathbb{R}^n$ is an arbitrary vector. Please give methods for the evaluation of Ab, A^{-1}, and $|A|$, all with the computational complexity of $O(n)$.

4. (*15 points*) For the system
$$u_t = -v_{xx}, \quad v_t = u_{xx},$$

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analyse the truncation error and stability of the scheme

$$\frac{u_j^{n+1}-u_j^n}{\tau} = -\frac{1}{2h^2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n + v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}),$$
$$\frac{v_j^{n+1}-v_j^n}{\tau} = \frac{1}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}).$$

5. (*15 points*) For the advection equation $u_t + au_x = 0$ (a is a constant), the Lax-Wendroff scheme reads as

$$u_j^{n+1} = -\frac{1}{2}\nu(1-\nu)u_{j+1}^n + (1-\nu^2)u_j^n + \frac{1}{2}\nu(1+\nu)u_{j-1}^n,$$

where $\nu = a\tau/h$.
(i) For the Cauchy problem imposed on the real line, show that

$$\|u^{n+1}\|_2^2 = \|u^n\|_2^2 - \frac{1}{2}\nu^2(1-\nu^2)\left(\|\delta_x^+ u^n\|_2^2 - \langle \delta_x^+ u^n, \delta_x^- u^n\rangle\right),$$

where

$$\|v\|_2^2 = \sum_j |v_j|^2, \quad \langle v, w\rangle = \sum_j v_j w_j, \quad \delta_x^- v_j = v_j - v_{j-1}, \quad \delta_x^+ v_j = v_{j+1} - v_j.$$

(ii) Suppose $a > 0$, for the problem imposed on $(0,1)$ with homogeneous boundary condition at $x = 0$ (i.e., $u_0^n = 0$), give a simple numerical boundary condition for $x = 1$ such that the Lax-Wendroff scheme is stable.

Remark: You can choose either 6 or 7. The points will be decided as max(6,7).

6. (*20 points*) Consider a harmonic oscillator with a cubic damping term

$$y'' + y + \epsilon(y')^3 = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

where $y = y(t)$, $t \geq 0$, $\epsilon > 0$.
(i) For small ϵ, use the multiple-scale method to study the behavior of $y(t)$ for large t, i.e., construct a proper asymptotic solution.
(ii) Make a conclusion on the validity of your asymptotic solution. Briefly justify your conclusion.

7. (*20 points*) Let K be a nonempty closed convex set in $\mathbb{R}^n$. For any $z \in \mathbb{R}^n$, define $\Pi_K(z)$ as the optimal solution of the problem $\min \frac{1}{2}\|y - z\|$, s.t. $y \in K$. Prove the following statements.
(i) Show that $\Pi_K(z)$ satisfies

$$\langle z - \Pi_K(z), d - \Pi_K(z)\rangle \leq 0, \quad \forall d \in K.$$

(ii) Define

$$\Theta(z) := \frac{1}{2}\|z - \Pi_K(z)\|^2.$$

Prove that $\Theta(\cdot)$ is continuous differentiable convex function and

$$\nabla\Theta(z) = z - \Pi_K(z).$$