

清华大学考试试题专用纸

考试课程 Subject: 求真书院代数博士生资格考 Algebra Qualifying Exam for Qiuzhen College

姓名 Name: _____

学号 Student ID: _____

- Exam Time: September 2026 (3 hours)
- This exam has 4 pages of 7 basic problems and 2 out of 6 advanced problems (100 points).
- Candidates are expected to adhere to the examination discipline by default, and those who do not comply will bear the consequences themselves.
- To receive full credit, all answers must provide necessary **details, proofs, and justifications**.
- You are strongly encouraged to write your answers **in English**.
- 第 8-13 题为提高题，你需要选择其中任意两题作答。请在答题纸首页清晰画出并填写下列表格，以注明需评分的两道题目。未在首页列出的题目将不予评分。

Q.8-13 are advanced problems and **selective**. Choose and answer **two** of them. No other answers will be graded. **Draw and fill the following table on the front page of the answer sheet to declare which two questions need to be graded.**

Advanced Problems		
Q.		
Score		

In the following problems, $\mathbf{Q}$ is the field of rational numbers, $\mathbf{R}$ is the field of real numbers, $\mathbf{C}$ is the field of complex numbers, $\mathbf{Z}$ is the ring of integers, $\mathbf{F}_q$ is the finite field consisting of q elements. **In all the problems, a ring means a commutative ring with multiplicative unit.**

1 基础题 Basic Questions

Q. 1 (10 points). *Classify all groups of order 99 up to isomorphism.*

Q. 2 (10 points). *Decide whether every 2×2 complex matrix has a complex matrix square root. Give a proof or a counterexample.*

Q. 3 (10 points). *Let k be a field and let $R = k[x]$. Suppose that $a, b \in R$ are nonzero polynomials, and set*

$$M = R/(a), \quad N = R/(b).$$

Prove that there exists $d \in R$ such that

$$\mathrm{Tor}_i^R(M, N) \cong \begin{cases} R/(d), & i = 0, 1, \\ 0, & i \geq 2. \end{cases}$$

Deduce that

$$\mathrm{Tor}_1^R(M, N) = 0$$

if and only if d is a unit in R .

Q. 4 (10 points). *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathfrak{gl}_4(\mathbf{C}).$$

Find all Borel subalgebras of $\mathfrak{gl}_4(\mathbf{C})$ that contain A .

Q. 5 (10 points). *Consider the polynomial*

$$f(X) = X^4 - 10X^2 + 20 \in \mathbf{Q}[X].$$

1. *Prove that f is irreducible over $\mathbf{Q}$.*

2. *Prove that the splitting field of f is $L = \mathbf{Q}(\alpha)$ where $\alpha = \sqrt{5 + \sqrt{5}}$.*

3. *Determine $\mathrm{Gal}(L/\mathbf{Q})$, and list all intermediate fields between $\mathbf{Q}$ and L .*

Q. 6 (10 points). *Let*

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, \quad (s_1 s_2)^4 = 1 \rangle.$$

Compute the character table of W .

Q. 7 (10 points). *Let k be a field, and let*

$$R = k[x, y]/(x, y)^2.$$

For each nonzero pair $(a, b) \in k^2$, let $M(a, b)$ be the two-dimensional R -module with k -basis e_1, e_2 whose R -module structure is given by

$$xe_1 = ae_2, \quad ye_1 = be_2, \quad xe_2 = ye_2 = 0.$$

1. Prove that $M(a, b)$ is indecomposable.
2. Show that the endomorphism ring $\text{End}_R(M(a, b))$ is isomorphic to $k[\varepsilon]/(\varepsilon^2)$ as a k -algebra.
3. Let (a, b) and (c, d) be both nonzero pairs in k^2 . Prove that

$$M(a, b) \cong M(c, d)$$

if and only if

$$(c, d) = \lambda(a, b)$$

for some $\lambda \in k^\times$.

4. Prove that every indecomposable R -module of k -dimension two is isomorphic to $M(a, b)$ for some nonzero pair $(a, b) \in k^2$.

2 高等题 Advanced Questions

Q. 8 (15 points). Find two sets of closed points $\{P_1, P_2, P_3\}, \{Q_1, Q_2, Q_3\} \subset \mathbf{A}^1$ such that

$$\mathbf{A}^1 \setminus \{P_1, P_2, P_3\} \not\cong \mathbf{A}^1 \setminus \{Q_1, Q_2, Q_3\}.$$

(Assume the base field is $\mathbf{C}$.)

Q. 9 (15 points). Determine whether the following projective plane curves are smooth, and compute the geometric genus of their normalizations. (Assume $\text{char } k = 0$ or $\text{char } k \geq 5$).

1. $YZ = X^2$.
2. $Y^2Z = X^3$.
3. $Y^2Z = X^3 - XZ^2$.

Q. 10 (15 points). Let A be a Dedekind domain. Denote by K the fraction field of A .

1. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be pairwise distinct maximal ideals of A . Let $e_1, \dots, e_n$ be elements of $\mathbf{Z}$. Show that there exists $x \in K$ such that in the factorization of the fractional ideal (x) , the exponent of $\mathfrak{p}_i$ is e_i for all $1 \leq i \leq n$.
2. Show that any ideal of A can be generated by two elements.

Q. 11 (15 points). Let ζ_7 be a 7-th primitive root of 1 and $K = \mathbf{Q}(\zeta_7)$. Let $q \in \mathbf{Z}$ be a prime number and let β be a maximal ideal of $\mathcal{O}_K$ above q . Determine the residue degree $f_{\beta/q}$.

Q. 12 (15 points). Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra. Fix a Cartan subalgebra and a Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}$ and $\mathfrak{n}^-$ respectively be the subalgebra spanned by positive and negative roots. Let $Z(U\mathfrak{g})$ be the center of the enveloping algebra $U\mathfrak{g}$. Show that

$$Z(U\mathfrak{g}) \cap (U\mathfrak{n} \cdot U\mathfrak{n}^-) = \mathbf{C} \cdot 1.$$

Q. 13 (15 points). Let $\mathcal{O}_0$ be the principal block of the BGG category $\mathcal{O}$ for $\mathfrak{g} = \mathfrak{sl}_3$.

- (1) Find the highest weights of all the indecomposable projective modules in $\mathcal{O}_0$.
- (2) For each module in (1), find its length.

You may use the following fact: each Kazhdan–Lusztig polynomial for S_3 is either zero or a monomial with coefficient 1.