

Select three out of four problems to work out. You must declare which three you choose, otherwise the first three will be graded.

## Classical Mechanics

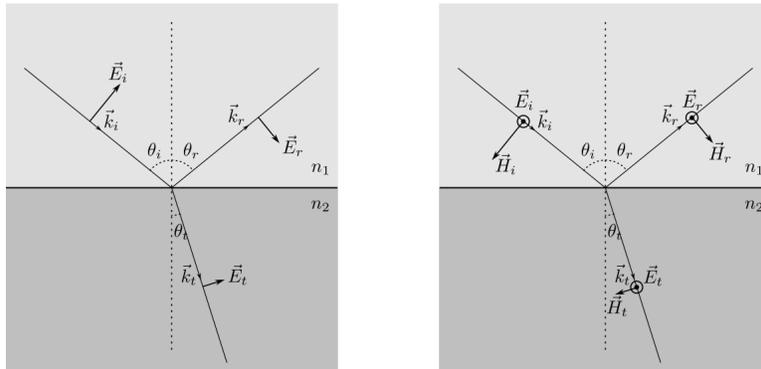
Consider a uniform solid half-cylinder of mass  $m$  and radius  $a$  rocking without slipping on a horizontal plane.

1. Determine the position of the center of mass of the half-cylinder and the moment of inertial  $I_{cm}$  around the center of mass.
2. Derive the Lagrangian  $L$  in terms of the angle  $\theta$  between the flat surface of the half-cylinder and the horizontal.
3. Write down the Lagrange equation of motion and find the frequency  $\omega$  of small oscillations around the equilibrium.
4. Determine the force  $\vec{F}(\theta)$  produced by the plane on the half-cylinder at the line of contact as a function of the angle  $\theta$  of the half-cylinder in the regime of small oscillations.

## Electrodynamics

Consider a plane electromagnetic wave incident on a boundary between two dielectric materials.

Reflection and Polarization



1. What components of the fields ( $\mathbf{D}$ ,  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{B}$ ) are continuous across the boundary? Hint: Consider the integral form of Maxwell's equations at the boundary.

- Using these relationships, find the ratio of the electric field amplitudes of electromagnetic plane waves incident and reflected from a planar boundary between two materials with indices of refraction given by  $n_1$  and  $n_2$  as a function of angle of incidence,  $\theta_i$ . Consider (p-)polarization where the electric field is in the plane of incidence/reflection (and the magnetic field is parallel to the boundary) as shown in the diagram on the left. You may find it helpful to use Snell's law ( $n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$ ), and you can assume that the two media have the same magnetic properties,  $\mu_1 = \mu_2$ . Also, you can make use of the relationship  $k = n\omega/c$  between the wave-number  $k$ , the frequency  $\omega$ , and the refraction index  $n$ , as well as the relationship between the field strengths in the plane waves,  $H = kE/(\mu\omega)$ .
- Find the reflection coefficient for the other (s-)polarization in which the magnetic field is in the plane of incidence/reflection (and the electric field is parallel to the boundary) as shown in the figure on the right.
- For what angle and polarization does the reflection coefficient vanish if  $n_1 = 1.2$  and  $n_2 = 1.5$ ?

## Quantum Mechanics

A particle of mass  $m$  and electric charge  $q$  moves in 1-dimension under the effects of a harmonic potential and a homogeneous electrostatic field  $E$ . The Hamiltonian for the system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - qEx = H_0 - qE\hat{x}. \quad (1)$$

- Show that  $H$  can be written as  $\hat{H} = e^{-\hat{A}}\hat{H}_0e^{\hat{A}} + B$  by explicitly determining the operator  $\hat{A}$  and the number  $B$ . Hint: recall  $e^{-ia\hat{p}/\hbar}\hat{x}e^{ia\hat{p}/\hbar} = \hat{x} - a$ .
- Use the above relation to show that the spectrum of  $\hat{H}$  follows from that of  $\hat{H}_0$  by a shift. Use this observation to solve the eigenvalue problem of  $\hat{H}$  in terms of  $\hat{H}_0$ .
- Express  $\hat{A}$  in terms of the creation  $a$  and annihilation  $a^\dagger$  operator of  $\hat{H}_0$ . Use this to evaluate the probability to find the system in the ground state of  $\hat{H}$  at time  $t$  if at time  $t = 0$  it is in the ground state of  $\hat{H}_0$  (But evolving according to  $\hat{H}$ ).
- What is the probability for the system to start at  $t = 0$  in the ground state of  $\hat{H}_0$  and remain in this state at time  $t$  (evolving according to  $\hat{H}$ )? For what time this probability is 1?
- Compute the expectation value of the dipole moment  $d = q\hat{x}$  at time  $t$ , assuming that at  $t = 0$  the system is again in the ground state of  $\hat{H}_0$  (evolving according to  $\hat{H}$ ).

## Statistical Mechanics

This problem addresses properties of the spontaneous electromagnetic radiation at thermal equilibrium.

1. Calculate the probability for a one-dimensional quantum harmonic oscillator of eigenfrequency  $\omega$  to be on its  $n$ -th energy level, in thermal equilibrium at temperature  $T$ .
2. Calculate the average energy, the free energy, and the entropy of the oscillator, and discuss their dependences on temperature.
3. Calculate the variance (dispersion) of fluctuations of oscillator's energy, and express it via the average energy and  $\hbar\omega$ .
4. Calculate the number of electromagnetic standing-wave modes in a large, closed free-space volume  $V$ , with frequencies within a narrow interval  $[\omega, \omega + d\omega]$ , where  $d\omega$  is much smaller than  $\omega$ , but still large enough to contain many modes. Each mode may be treated as a one-dimensional quantum harmonic oscillator.
5. Calculate the average total energy of the electromagnetic field in volume  $V$  (including all essential modes), and the variance of its fluctuations. Express the variance via the average energy and temperature, and find the dependence of the relative r.m.s. fluctuation of the energy on temperature  $T$  and volume  $V$ .

Hint: You may like to use the following integral:

$$\int_0^{\infty} \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \quad (2)$$