Classical Mechenics (25 pts)

Consider a point particle with mass m, moving on a fixed sphere with radius R with gravity constant g.

1. Write down the Lagrangian in spherical coordinates (r, θ, ϕ). Write down explicit expressions for the two conserved quantities.

2. Write down the Hamiltonian in spherical coordinates (r, θ, ϕ) and their corresponding conjugate momenta. Show that the Poisson bracket of the above two conserved quantities is 0.

3. Using the expressions in question 1, write the time t as an integral of θ, and ϕ as an integral of θ.

4. If the particle moves on a horizontal circle with azimuthal angle θ, what is the angular velocity of this periodic orbit?

5. If the particle has a z-component M_z of the orbital angular momentum, all its trajectories will lie between two horizontal circles on the sphere. Find an equation for the position of these circles. How many solutions does this equation have? Which solutions correspond to the positions of these two circles?

Quantum Mechanics (25 pts)

Consider a particle on the x-axis with potential $U(x)$ such that $U(x)$ vanishes as $x \to \pm\infty$, and $U(x)$ is everywhere negative and smooth. Recall that the ground state for such a system is always a nondegenerate bound state.

1. Define $V(x) = U(x) - E_0$ where E_0 is the grouund state energy. Write the Hamiltonian in factorized form as $H = A^\dagger A + E_0$, where $A = c\frac{d}{dx} + W(x)$ and c is a constant. Determine c and $W(x)$. (Hint: Express $V(x)$ in terms of the ground state wave function $\phi_0(x)$ and then try to express W in terms of $\phi_0(x)$.)

2. Show that A annihilates $\phi_0(x)$.

3. Show that $H_1 = A^\dagger A + E_0$ and $H_2 = AA^\dagger + E_0$ have the same eigenvalues $E_n > E_0$. Given an eigenvector $|E_n\rangle$ of H_1 with eigenvalue $E_n > E_0$, construct an eigenvector of H_2 with the same eigenvalue.

4. For $H_1 = A^\dagger A + E_0$ and $H_2 = AA^\dagger + E_0$. Let $A^\dagger A = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1$ and $AA^\dagger = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2$. Construct $W(x)$ which gives $V_1 = a^2$, where a is a constant, and construct the corresponding $V_2(x)$.

5. If $A^\dagger A$ has a constant potential $V_1 = a^2$, the eigenfunctions for H_1 are plane waves. Construct eigenfuctions of H_2 using these eigenfunctions

of H_1. Show these eigenfunctions have no reflection from $V_2(x)$, i. e. for every incoming plane wave of the continuous spectrum there is only transmission wave.

Thermodynamics (25 pts)

Consider an ideal 3d gas of N ultra-relativistic electrons (spin 1/2) with energies $E_{\vec{p}} = |\vec{p}|c$ (where $\vec{p}$ is electron's momentum, and c the speed of light), confined to volume V

1. For the gas in equilibrium at 0 temperature, calculate its chemical potential μ (i.e. the Fermi energy E_F) and the total energy E_0, and express E_0 in terms of N and E_F.

2. Now consider the gas in equilibrium at a low temperature $T \ll E_F/k_B$. In the first nonvanishiong approximation in T, calculate the chemical potential, and express your result in terms of E_F and T.

3. Again at low temperature $T \ll E_F/k_B$ as in question 2, calculate the specific heat (i.e. the heat capacity per particle) of the gas, and express it in terms of E_F and T.

4. Obtain general expressions for the grand thermodynamic potential of the gas and its pressure, and express them via the total energy of the gas and its volume.

5. Express the gas pressure at $T = 0$ in terms of N and V.

Hint 1: You may find the following Sommerfeld expansion useful

$$\int_0^\infty F(E)f(E)dE = \int_0^\mu F(E)dE + \frac{\pi^2}{6}(k_B T^2)F'(\mu) + O\left(\left(\frac{k_B T}{\mu}\right)^4\right), \quad (1)$$

where

$$f(E) = \frac{1}{e^{(E-\mu)/(k_B T)} + 1}, \quad (2)$$

is the Fermi-Dirac distribution, $F(E)$ is any differentiable function, growing slower than $1/f(E)$ at $E \to \infty$ and $F'(E)$ is its derivative.

Hint 2: The grand thermodynamic potential for each state of a Fermi gas is

$$\Omega(E) = -k_B T \ln(1 + e^{(\mu - E)/(k_B T)}). \quad (3)$$

If the total grand thermodynamic potential of the system is Ω, then $P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T,N}$.

Electrodynamics (25 pts)

A uniform conducting sphere of radius a, with electric and magnetic permeabilities $\epsilon = \mu = 1$ and conductivity σ, rotates with constant angular velocity ω around the z-axis. A uniform magnetic field of magnitude B is applied along the axis of rotation. The initial charge on the sphere is zero. Ignoring the magnetic field due to the rotating sphere, evaluate in the steady state:

1. The electric field in the sphere. (Hint: in equilibrium, the free charges (electrons) in the sphere redistribute because of the Lorentz force)

2. The volume charge density inside the sphere.

3. The electric potential and field inside and outside the sphere. Hint: when $r > a$ the potential with cylindrical symmetry can be expanded in terms of Legendre polynomials

$$\phi(r > 0) = \sum_{n=0}^{\infty} C_n \frac{a^n}{r^{n+1}} P_n(\cos\theta). \tag{4}$$

4. The charge density on the surface of the sphere.

Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x). \tag{5}$$