General Relativity

Instructions: Answer all questions. Show all steps and justify all assumptions. Partial credit may be awarded for clear demonstration of understanding, even if the final calculation is incomplete. State any external theorems or physical principles you use.

Constants: Speed of light $c = 1$.

Question 1 (The Schwarzschild Black Hole) (25 marks)

The Schwarzschild metric describes the spacetime outside a spherically symmetric, non-rotating mass M:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Consider motion in the equatorial plane ($\theta = \pi/2$).

(a) Write down the conserved quantities associated with the Killing vectors $\xi_{(t)} = \partial_t$ and $\xi_{(\phi)} = \partial_\phi$. Define E (energy per unit mass) and L (angular momentum per unit mass) from these quantities.

(b) Starting from the condition $g_{\mu\nu}u^\mu u^\nu = -1$ for a massive particle (or $= 0$ for a photon), derive the radial "energy equation":

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}}(r) = \frac{1}{2}\tilde{E}^2$$

where λ is proper time τ for a massive particle or an affine parameter for a photon. Find the form of the effective potential $V_{\text{eff}}(r)$ in terms of r, L, and M. Clearly indicate how your expression differs for massive particles and photons. (You may use the notation $\tilde{E} = E/m$ and $\tilde{L} = L/m$ for massive particles, and simply E, L for photons.)

(c) **Infall Time:** Consider a massive test particle with $L = 0$ (radial infall) and $E = 1$ (starting from rest at infinity). Calculate the proper time $\Delta\tau$ it takes for the particle to fall from the event horizon at $r = 2M$ to the singularity at $r = 0$.

Question 2(The Newtonian Limit) (25 marks)

A weak gravitational field can be described by a perturbation to flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

(a) Show that in this limit, the geodesic equation for a slowly moving particle ($dx^i/d\tau \ll dt/d\tau$) reduces to

$$\frac{d^2x^i}{dt^2} = -A\partial_i h_{00}.$$

Determine the constant A.

(b) By comparing this result with the Newtonian equation of motion $\ddot{\mathbf{x}} = -\nabla\Phi$, where Φ is the Newtonian gravitational potential, identify the relation between h_{00} and Φ.

(c) Using the relation from (b) and the fact that the Newtonian potential satisfies Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, what would you guess the 00-component of the Einstein field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ should reduce to in the weak-field limit? Briefly justify your answer.