Name: ________________ Student ID: ________________

Qualifying Exam-Analysis

Fall 2024

1. (10 pts) Find all the conformal maps from $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$. Verify your answer.

2. (10 pts) Let $\Omega \subset \mathbb{C}$ be a simply connected domain and u a real-valued harmonic function on Ω. Prove that up to a constant, there exists a unique holomorphic function f on Ω such that $Re(f) = u$.

3. (10 pts) Find all $a \in \mathbb{R}$ such that the following boundary value problem has no solution $u = u(x)$:
$$\frac{d^2u}{dx^2} + au = 1, \quad u(0) = u(1) = 0.$$

4. (10 pts) Let $u \in \mathscr{S}'(\mathbb{R}^3)$ be the standard surface measure on the unit sphere
$$\mathbb{S}^2 = \{x = (x_1, x_2, x_3) : |x|^2 = x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3.$$
Compute its Fourier transform $\hat{u}(\xi)$ for $\xi \in \mathbb{R}^3$. (Here we use the following definition for Fourier transform of L^1 functions
$$\hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x) dx$$
and extend to tempered distributions.)

5. (15 pts) Consider the following operator
$$Tf(x) = \int_0^\infty \frac{f(y)}{x+y} dy,$$

(i) Show that for $p \in (1, \infty)$, and any $f \in C_c([0, \infty))$,

$$\|Tf\|_{L^p([0,\infty))} \leq C_p \|f\|_{L^p([0,\infty))},$$

where

$$C_p := \int_0^\infty \frac{dx}{x^{1/p}(1+x)}.$$

(ii) Compute C_p.

(iii) Show that the result in (i) does not hold for any $C_p > 0$ when $p = 1$ or $p = \infty$.

6. (15 pts) Find all C^1 solutions $u = u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ of the equation

$$|\nabla u| := \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} = 1.$$

7. (15 pts) Let X be a Banach space, and $V \subset X$ be a subspace with finite codimension.

(i) Show that V is not necessarily closed by providing a counter example.

(ii) Suppose there exist a Banach space Y and a bounded linear operator A from Y to X, such that $V = R(A)$. Show that V is closed.

8. (15 pts)

(i) Let θ: $\mathbb{R} \to \mathbb{R}^+$ be a C^1 function such that $\|\theta'/\theta\|_{L^\infty(\mathbb{R})} \leq 1$. Show that for all $u \in C_c^\infty(\mathbb{R})$, there exists some absolute constant C s.t.,

$$\int_{\mathbb{R}} u^6 \theta \leq C \|u\|_{L^2(\mathbb{R})}^4 \int_{\mathbb{R}} (|u'|^2 + |u|^2)\theta.$$

(ii) Let ω: $\mathbb{R}^2 \to \mathbb{R}^+$ be a C^1 function such that $\|\nabla\omega/\omega\|_{L^\infty(\mathbb{R}^2)} \leq 1$. Show that for all $u \in C_c^\infty(\mathbb{R}^2)$, there exists some absolute constant C s.t.,

$$\int_{\mathbb{R}^2} u^4 \omega \leq C \|u\|_{L^2(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2)\omega.$$