# Qualifying Exam: 2024 Spring

Concentration: Probability & Statistics     Name: ____________     Student ID: ____________

- There are 11 problems in this exam (2 pages). You choose 8 of them to finish. If you select more than 8, only the first 8 that you have worked on will be graded. Note that 4 of the problems are worth 15 points each and the rest 10 points each.
- You must follow all the rules of exam taking. Misconducts will be subject to proper disciplinary actions by the Center.
- You must provide all necessary details for full credits. A final answer with no or little explanation/derivation, even if correct, receives a minimal credit.

Note: $\ln x := \log_e x$. $\mathbb{R}$ represents the set of all real numbers. $\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the set of all positive integers.

1. (10 points) Let $X_1, X_2, \dots$ be independent identically distributed random variables with density

$$f(x) = \begin{cases} |x|^{-3}, & |x| > 1 \\ 0, & \text{otherwise} \end{cases}$$

and characteristic function $\phi(t) := \mathbb{E}e^{itX_1}$. Let $S_n := X_1 + X_2 + \cdots + X_n$ for each $n \in \mathbb{N}$.

   (a) Prove that $\mathbb{E}(X_1^2) = \infty$.

   (b) It is known that $\lim_{t\to 0} \frac{1-\phi(t)}{t^2 \ln|t|} = -1$; you do not need to prove this. Using this fact, find a sequence $(a_n)_{n\in\mathbb{N}}$ and prove that $S_n/a_n$ converges in distribution to the standard normal distribution (i.e., mean 0 and variance 1).

2. (15 points) Let $X_1, X_2, \dots$ be independent identically distributed random variables with exponential distribution: $\mathbb{P}(X_k > x) = e^{-x}$ for $x \geq 0$. Let $M_n := \max_{1\leq k\leq n} X_k$. Show that

   (a) $\limsup_{n\to\infty} X_n/\ln n = 1$ almost surely.

   (b) $\lim_{n\to\infty} M_n/\ln n = 1$ almost surely.

3. (10 points)

   (a) Prove that: if $X_n \xrightarrow{\text{P}} X$ (i.e., $X_n$ converges to $X$ in probability) and $Y_n \xrightarrow{\text{P}} Y$ then $X_nY_n \xrightarrow{\text{P}} XY$.

   (b) Is the following true or false: if $X_n \xrightarrow{L^1} X$ (i.e., $X_n$ converges to $X$ in $L^1$) and $Y_n \xrightarrow{L^1} Y$ then $X_nY_n \xrightarrow{L^1} XY$? Either prove it or give a counter example.

4. (15 points) Let $p \in (0, 1)$ and $k \in \mathbb{N}$. Let $X_1, X_2, \cdots$ be independent random variables with $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = -1)$ for each $n \in \mathbb{N}$. Let $X_0 := 0$ and $S_n := X_0 + X_1 + \cdots + X_n$ for each $n \in \mathbb{N} \cup \{0\}$. Let $T_k := \inf\{n \in \mathbb{N} : S_n = k\}$ with $\inf \emptyset = \infty$. For each $t \in \mathbb{R}$, let $M(t) := \mathbb{E}e^{tX_1}$.

   (a) If $p \in [1/2, 1)$, prove that $e^{tk}\mathbb{E}M(t)^{-T_k} = 1$ for all $t > 0$.

   (b) If $p \in (0, 1/2)$, compute $\mathbb{P}(T_k < \infty)$.

(c) If $p \in (0, 1/2)$, find the distribution of $Y := 1 + \sup_{n \geq 0} S_n$.

5. (10 points) Let $S$ be a countable state space and $(Z_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables taking values in a measurable space $(E, \mathcal{E})$.

   (a) Suppose that $f : S \times E \to S$ is a measurable mapping and $X_0$ is independent of all $Z_n$. Let
   $$X_n := f(X_{n-1}, Z_n), n \in \mathbb{N}. \tag{1}$$
   Prove that $(X_n, n \in \mathbb{N} \cup \{0\})$ is a Markov chain, and determine its transition probability.

   (b) Prove that any (discrete) time-homogeneous Markov chain on $S$ can be written in the form (1).

6. (10 points) Let $(B_t, t \geq 0)$ be a one-dimensional Brownian motion starting from the origin (i.e, $B_0 = 0$). Let $\mathcal{F}_t := \sigma(B_s : s \leq t)$ be the filtration generated by $B_t$. For $a > 0$, define $T_a := \inf\{t > 0 : B_t \geq a\}$.

   (a) Prove that $T_a$ is a $\mathcal{F}_t$ stopping time.

   (b) Compute $\mathbb{E}e^{-\lambda T_a}$ for $\lambda > 0$, and prove that $\mathbb{P}(T_a < \infty) = 1$. *Hint:* one may consider $M_t := e^{uB_t - u^2 t/2}$.

7. (15 points) Let $X_i$ be independent $\mathrm{N}(\mu_i, 1)$, $i = 1, ..., n$. Let $\mu = (\mu_1, ..., \mu_n)^T$, $\mathbf{X} = (X_1, ..., X_n)^T$, $\mathbf{a} = (a_1, ..., a_n)^T$. Consider the following loss function
$$l(\mu, \mathbf{a}) = \sum_{i=1}^{n} (a_i - \mu_i)^2,$$
prove that $\delta(\mathbf{X}) = \mathbf{X}$ is minimax.

8. (10 points) Suppose $X_1, X_2, ..., X_n$ are i.i.d. random variables from $Uniform(\theta, \theta + 10)$ distribution with $\theta > 0$.

   (a) Prove that $\hat{\theta} = \min(X_1, ..., X_n)$ is a consistent estimator of $\theta$.

   (b) Find the asymptotic distribution of
   $$\sqrt{12n} \left( \frac{\overline{X} - 5 - \theta}{\hat{\theta}} \right),$$
   where $\overline{X}$ is the sample mean and $\hat{\theta}$ is the consistent estimator for $\theta$ in part (a).

9. (10 points) Let $(X_1, ..., X_n)$ be a random sample and assume that $\log X_i$ is distributed as $\mathrm{N}(\theta, \theta)$ with an unknown parameter $\theta > 0$.

   (a) Derive the MLE of $\theta$, $\hat{\theta}$.

   (b) What is the asymptotic distribution of $\hat{\theta}$?

10. (10 points) Let $X_1, ..., X_n$ be the incomes of $n$ persons chosen at random from a certain population. Assume that $X_i$ has the following Pareto density
$$f(x \mid \theta) = 100^{\theta} \theta x^{-(1+\theta)}, x > 100,$$
where $\theta > 1$. Let $\mu = \mathrm{E}(X)$ be the mean of the income, write down the rejection region of the UMP level $\alpha$ $(0 < \alpha < 1)$ test of testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$.

11. (15 points) Let $\theta_1$, $\theta_2$ and $\theta_3$ be nonnegative parameters with the constraint $\theta_1 + \theta_2 + \theta_3 = 1$. We observe $X_{i1} = \theta_1 + \epsilon_{i1}$, $X_{i2} = \theta_2 + \epsilon_{i2}$, $X_{i3} = \theta_3 + \epsilon_{i3}$ for $i = 1, 2, ..., n$, where $\epsilon_{ik} \sim N(0, 1)$ are independent normal random variables $(k = 1, 2, 3)$. Derive the uniformly minimum-variance unbiased estimator (UMVUE) for $\theta_1$.