清华大学考试试题专用纸

姓名: ____________ 学号: ____________

Everyone needs to solve Problem 1, 2, 3, 4 (10 points each). Please select four problems in the following table and solve them (15 points each).

Problem 5	Problem 6	Problem 7	Problem 8	Problem 9	Problem 10

1. Let M, N be closed, connected manifolds of **the same dimension**. Suppose $f : M \to N$ is an immersion. Is the following true or false? If true, prove your claim. If false, give a counterexample and prove that your example is indeed a counterexample.

   (a) f is injective.

   (b) f is surjective.

   (c) f is a homeomoprhism.

   (d) f is a covering map.

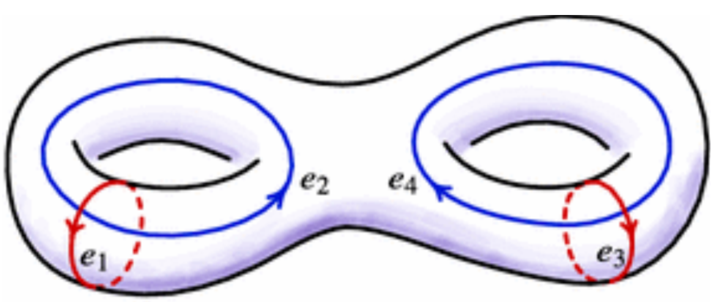
2. Let $\mathbb{B}^n(r) = \{x \in \mathbb{R}^n \mid |x| < r\}$. Compute the sectional curvature of

$$g = \frac{r^4}{(r^2 - |x|^2)} \sum_{i=1}^{n} dx^i \otimes dx^i.$$

3. Let X denote a closed orientable surface of genus 2. See the picture below with four curves labeled as e_1, e_2, e_3, e_4. Consider a group homomorphism

$$\rho : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$$

   such that $\rho(e_4) = 1$ and $\rho(e_i) = 0$ for $i = 1, 2, 3$.



   (a) Draw a picture of a covering space Y that corresponds to the subgroup $\ker \rho \subset \pi_1(X)$.

   (b) On the picture, label all curves on Y which are preimages of e_1, e_2, e_3, e_4.

(c) The signature $\sigma(A)$ of a square matrix A is the number of its positive eigenvalues minus the number of its negative eigenvalues. Let T denote the nontrivial deck transformation of the covering $Y \to X$. Compute the signature $\sigma(T_*)$ of $T_* : H_1(Y) \to H_1(Y)$.

(d) Prove that there exists a homeomorphism $\phi : X \to X$ such that $\phi(e_1) = e_4$.

4. Let (M, g) be a Riemannian manifold and X be a Killing field. If γ is a geodesic, show that $J(t) = X \circ \gamma(t)$ is a Jacobi field along γ.

5. Compute the **cohomology ring** of $H^*((S^2 \times S^8)\#(S^4 \times S^6); \mathbb{Z})$.

6. A Riemannian metric h on a Lie group G is said to be **left-invariant** if

$$L_g^* h = h$$

for all $g \in G$. A **right-invariant metric** is defined similarly. A metric that is both left- and right-invariant is said to be **bi-invariant**. Let (M, g) be a Riemannian manifold with a bi-invariant Riemannian metric g.

(A) Show that for all left-invariant vector fields X, Y, Z,

$$h([X, Y], Z) = h(X, [Y, Z]).$$

(B) Show that for any left-invariant vector fields X, Y,

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

wehre ∇ is the Levi-Civita connection.

7. Suppose that M is a compact connected **nonorientable** 3-dimensional manifold. Prove that $\pi_1(M)$ is infinite.

8. Let (M, g) be a **simply connected** complete Riemannian manifold. If the differential of each exponential map is length increasing, i.e.

$$\left| \left(d \exp_p\right)_v \tilde{v} \right| \geq |\tilde{v}|$$

for all $p \in M$ and all $v, \tilde{v} \in T_p M$, show that (M, g) has non-positive sectional curvature.

9. Let $SL_3(\mathbb{C})$ be the space of 3×3 complex matrices with determinant 1. Compute $\pi_3(SL_3\mathbb{C})$.

10. Let (M, g) be an n-dimensional compact oriented Riemannian manifold with positive sectional curvature. Given an isometry $F : M \to M$ such that F preserves the orientation when n is even, F changes the orientation when n is odd. Show that F has a fixed point.