Name: ________________ Student ID: ________________

Qualifying Exam-Analysis

Fall 2023

1. (10 pts) For all $\xi \in \mathbb{R}$, compute
$$\int_{\mathbb{R}} \frac{e^{-ix\xi}}{(1+x^2)^2} dx.$$

2. (10 pts) Find all equilibrium points of the following system of ordinary differential equations for $(x(t), y(t))$ and determine their stability (stable or unstable):
$$\frac{dx}{dt} = \log(y^2 - x); \quad \frac{dy}{dt} = x - y - 1.$$

3. (10 pts) Find all distributions u on $\mathbb{R}^2$ such that
$$x_1 x_2 u = (x_1^2 - x_2^2)u = 0.$$

4. (10 pts) Consider the initial value problem for Burgers' equation
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}$$
$$u(0, x) = -x + x^3, \quad x \in \mathbb{R}.$$
Determine $T_{\max} > 0$ such that the problem has a C^1 solution in $(0, T_{\max}) \times \mathbb{R}$, but not in $(0, T) \times \mathbb{R}$ for any $T > T_{\max}$.

5. (15 pts) Determine whether the following statement is true or not: If $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then their convolution $f * g$ is a continuous function on $\mathbb{R}$ and satisfies
$$\lim_{|x|\to\infty} f * g(x) = 0.$$
Prove the statement if it is true and give a counterexample otherwise.

6. (15 pts) Let $\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disk and $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $g(0) = 0$ and $|\operatorname{Re} g(z)| < 1$ for any $z \in \mathbb{D}$. Prove that for any $z \in \mathbb{D}$,
$$|g(z)| \le \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}.$$

7. (15 pts) Let $\mathcal{H}$ be a complex Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, $\sigma(T)$ be the spectrum of T. We also define the numerical range of T to be

$$N(T) := \{(\psi, T\psi) | \psi \in \mathcal{H}, \|\psi\| = 1\} \subset \mathbb{C}$$

where $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and the norm on $\mathcal{H}$, respectively.

(1) Show that $N(T)$ is a convex set in $\mathbb{C}$.

(2) For any $\psi \in \mathcal{H}$ and $z \in \mathbb{C}$, show that

$$\|(T - z)\psi\| \geq d(z, N(T))\|\psi\|.$$

Here $d(z, N(T))$ is the distance from z to $N(T)$ in $\mathbb{C}$.

(3) Let $\overline{N(T)}$ be the closure of $N(T)$ in $\mathbb{C}$, show that $\sigma(T) \subset \overline{N(T)}$ and for any $z \in \mathbb{C} \setminus \overline{N(T)}$,

$$\|(T - z)^{-1}\| \leq d(z, N(T))^{-1}.$$

Here the norm on the left-hand side is the operator norm for bounded linear operator on $\mathcal{H}$.

8. (15 pts) Consider the following initial value problem for the heat equation in $\mathbb{R}^{1+3}$:

$$\frac{\partial u}{\partial t} - \Delta u = \mathcal{N}(u); \quad t > 0, x \in \mathbb{R}^3$$

$$u(0, x) = f(x) \in \mathcal{S}(\mathbb{R}^3).$$

Here $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz class of functions on $\mathbb{R}^3$.

(1) If $\mathcal{N}(u) \equiv 0$, show that the unique solution $u = u(t, x)$ in $C([0, \infty); \mathcal{S}(\mathbb{R}^3))$ satisfies

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(f)(1 + t)^{-3/2}$$

where $C(f)$ is a constant depending only on $f \in \mathcal{S}(\mathbb{R}^3)$.

(2) If $\mathcal{N}(u) = u^4$, show that there exists $\varepsilon_0 > 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$ with

$$\sup_{\alpha, \beta \in \mathbb{N}^3, |\alpha| + |\beta| \leq 1000} \sup_{x \in \mathbb{R}^3} |x^\alpha \partial^\beta f(x)| \leq \varepsilon_0$$

there is a solution $u = u(t, x) \in C^1((0, \infty); H^{10}(\mathbb{R}^3))$.