QFT problems for qualifying exam

February 21, 2026

1) Quantizing the Dirac Field

The Dirac Lagrangian is given by $\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ and the standard mode expansion for the field operator is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b_p^s u^s(p) e^{-ip.x} + d_p^{s\dagger} v^s(p) e^{ip.x} \right).$$

- Quantize the Dirac field in a box of volume V and show that the vacuum energy E_0 is indeed proportional to V.
  Hint: The integral over momentum $\int d^3p$ is replaced by a sum over discrete values of the momentum.
- Use Noether's theorem to derive the conserved current $J^\mu = \bar{\psi}\gamma^\mu\psi$. Calculate $[Q, \psi]$, thus showing that b and $d^\dagger$ must carry the same charge.

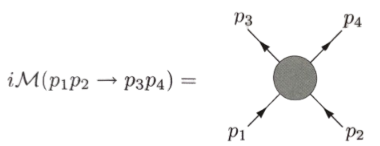
25 pts

2) Coupling constant renormalization in φ^4-theory

The Lagrangian of φ^4-theory with the counterterm for the interaction is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m_P^2 \varphi^2 - \frac{\lambda_P}{4!}\varphi^4 + C\varphi^4.$$

The two-particle scattering amplitude in φ^4-theory is given by



$$= -i\lambda_P + (-i\lambda_P)^2 \left[iV(s) + iV(t) + iV(u) \right] + 4! iC,$$

where the last term arises from the counterterm. The renormalization condition is given by

$$i\mathcal{M} = -i\lambda_P \quad \text{at} \quad s = 4m_P^2,\ t = u = 0.$$

Compute $V(p^2)$ using dimensional regularization and Feynman parameters. Fix C to second order in λ_P this way.
Hint: In Pauli-Villars regularization one replaces the propagator $\Delta(k, m^2)$ with the regulated version:

$$\frac{1}{k^2 - m^2 + i\epsilon} \to \frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon}.$$

25 pts