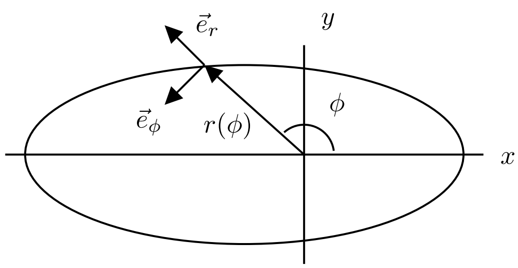
Select three out of four problems to work out. You must declare which three you choose, otherwise the first three will be graded.

Classical Mechanics

Consider a point particle with mass m and negative total energy $E = -|E|$ moving in the following potential

$$V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2} \quad (\alpha > 0, \beta \geq 0). \tag{1}$$

1. First prove that $\vec{e}_r = -\frac{d\vec{e}_\phi}{d\phi}$, where $\vec{e}_r$ is the unit vector along the radius and $\vec{e}_\phi$ the unit vector orthogonal to $\vec{e}_r$ in the direction of increasing ϕ (see figure).



   When $\beta = 0$, also prove that

$$\frac{d\vec{v}}{d\phi} = \gamma \frac{d\vec{e}_\phi}{d\phi}, \tag{2}$$

   where $\vec{v}$ is the velocity of the particle. What is the constant γ?

2. Assuming $\beta = 0$, it follows from (2) that $\vec{v}(t) = \vec{w} + \gamma \vec{e}_\phi(t)$ with constant $\vec{w}$. By taking the scalar product of this equation with $\vec{e}_\phi$, show that the orbit $r(\phi)$ is elliptical. (Hint: express $v_\phi \equiv \vec{v}(t) \cdot \vec{e}_\phi(t)$ in terms of polar coordinates. You may choose $\vec{w}$ along the y axis.)

3. On the other hand (still assuming $\beta = 0$), show that $\vec{v}(\phi)$ describes a circle.

4. Now consider the case that $\beta > 0$. Derive the relation

$$\frac{1}{2} m\dot{r}^2 + (\frac{A}{r} - B)^2 = \mathbf{E}^2. \tag{3}$$

   What are A, B and $\mathbf{E}$? Set $\sqrt{\frac{m}{2}}\dot{r} = \mathbf{E} \sin f(t)$ and $(\frac{A}{r} - B) = \mathbf{E} \cos f(t)$. Is this always possible?

5. Show that $\frac{f(t)}{\phi} = \frac{df}{d\phi} = \omega =$ constant. Show that the orbits $r(\phi)$ are now ellipses with precession (进动).

Quantum Mechanics

Consider the spinless particle with charge e and mass m in a constant magnetic field B directed along the z-axis

1. Let $\vec{A}$ be the vector potential corresponding to the magnetic field B. The operator of particle velocity is

$$\vec{v} = \frac{1}{m}(\vec{p} - \frac{e}{c}\vec{A}). \tag{4}$$

   Write down the non-relativistic Hamiltonian describing this problem. (Hint: $H = T + V$)

2. Establish the commutation relations for the spatial components of these operators $[v_i, v_j]$ and for $[v_i, x_j]$, where x_j are the coordinate operator. Explain what these commutators imply about measurements of these observables of the system.

3. Write down the Schroedinger equation describing the problem and find the energy spectrum (Landau levels). Write down the wave function of the lowest energy level. Finally determine the degeneracy of this energy level if the particle is confined in a rectangular region of area $A = a_x a_y$ perpendicular to the magnetic field.

4. Evaluate the commutator of the angular momentum component L_z and velocity component v_z. Is L_z conserved?

Thermodynamics

An external magnetic field B is applied to a set of N non-interacting spin-$\frac{1}{2}$ particles with gyromagnetic ratio (i.e. the ration between magnetic moment and angular momentum) γ, and fixed spatial positions. For the thermal equilibrium at temperature T, calculate:

1. The average energy and heat capacity per particle.

2. The average magnetic moment of the system and the variance of its fluctuations.

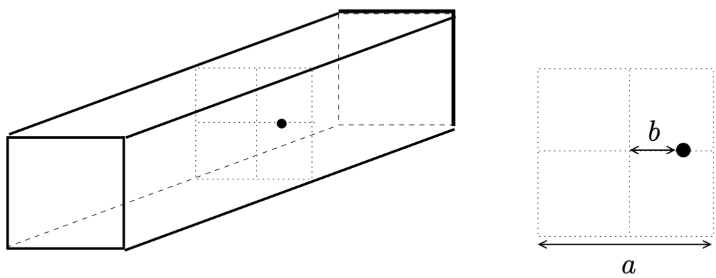
3. The entropy per particle.

4. Sketch the temperature dependence of the entropy, for both very large and very small field magnitudes, and discuss (qualitatively) what would happen with the entropy and the temperature of the system if it is first thermally isolated from the environment, and then the applied field is turned off gradually (you may assume this process is adiabatic and reversible in the first approximation).

5. Suggest a way to use this system as a refrigerator, assuming that its thermal contacts with hot and cold heat baths, and the applied magnetic field, may be controlled at will.

Electrodynamics

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.



1. Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x z)\Phi(k_y y)e^{\pm\kappa_z z}, \quad \text{where} \quad \Phi(u) = \{\cos u \text{ or } \sin u\}. \tag{5}$$

   for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

2. Now consider a point charge q displaced from the center of the tube by a distance b in the x direction, i.e. the coordinates of the charge are $\vec{r}_o = (x, y, z) = (b, 0, 0)$ ($q\delta(x - b)\delta(y)\delta(z)$ on the right-hand-side of the Poisson equation). Use the method of images to determine the potential.

3. As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (1) to determine the potential from the point charge described in part (2). (Hint: you may first show that for a suitable normalization $A\sum_n X_n(x)X_n(x_0) = \delta(x - x_o)$.)

4. Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$.