

## Geometry & Topology

### Syllabus

#### Smooth Manifold:

- Basics of smooth manifolds: Inverse function theorem, implicit function theorem, submanifolds, Sard's Theorem, embedding theorem, transversality, degree theory.
- Basics of matrix Lie groups over  $\mathbb{R}$  and  $\mathbb{C}$ : The definitions of  $GL(n)$ ,  $SU(n)$ ,  $SO(n)$ ,  $U(n)$ , their manifold structures, Lie algebras, right and left invariant vector fields and differential forms, the exponential map.
- Definition of real and complex vector bundles, tangent and cotangent bundles, basic operations on bundles such as dual bundle, tensor products, exterior products, direct sums, pull-back bundles.
- Definition of differential forms, exterior product, exterior derivative, Stokes theorem, de Rham cohomology, Hodge decomposition theorem, integration on manifolds.
- Definition of a principal Lie group bundle for matrix groups.
- Associated vector bundles: Relation between principal bundles and vector bundles.

#### Differential Geometry:

- Moving frame method, first and second fundamental forms, Gauss curvature, Gauss's Theorema Egregium, Gauss-Bonnet theorem.
- Riemannian metrics, definition of a geodesic, existence and uniqueness of geodesics.
- Definition of covariant derivative for a vector bundle and connection on a principal bundle. Relations between the two.
- Definition of curvature, flat connections, parallel transport.
- Definition of Levi-Civita connection and properties of the Riemann curvature tensor, manifolds of constant sectional curvature.
- First and second variation of length functional, Jacobi Fields.
- Bonnet-Meyers theorem, Cartan-Hadamard theorem, Synge's theorem, Gauss-Bonnet theorem.
- Comparison results in Riemannian geometry: Rauch comparison theorem, Hessian comparison theorem, Laplacian comparison theorem, etc.
- Manifolds of nonpositive curvature, manifolds of positive curvature.

#### Algebraic Topology:

- Fundamental groups, Seifert-Van Kampen theorem.
- Covering spaces, universal covering spaces, regular covering spaces, classification theorem of covering space.
- Classification of closed surfaces.
- Singular homology and cohomology, relative homology.
- CW complexes and the homology of CW complexes.
- Mayer-Vietoris sequence.
- Universal coefficient theorem and Kunneth formula.
- Cup product and cap product, cohomology ring.
- Fundamental class of manifolds, Poincare duality theorem, Lefschetz duality theorem, intersection number of submanifolds.

- Lefschetz fixed point formula.
- Hopf index theorem.
- Cech cohomology and de Rham cohomology.
- Equivalence between singular, Cech and de Rham cohomology.
- Higher homotopy groups (definition and basic properties).
- Fibrations and the long exact sequence of a fibration.
- Hurewicz theorem and Whitehead theorem.

### Written examination

#### Reference:

#### Smooth manifolds:

- [1] V. Guillemin, A. Pollack: Differential topology
- [2] J. Milnor: Topology from the differentiable viewpoint
- [3] John Lee: Introduction to Smooth manifold

#### Differential Geometry:

- [4] Cliff Taubes: Differential geometry: Bundles, Connections, Metrics and Curvature
- [5] John Lee: Introduction to Riemannian manifolds, second edition
- [6] Peter Petersson: Riemannian geometry
- [7] Do Carmo: Riemannian Geometry
- [8] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry

#### Algebraic Topology:

- [9] 姜伯驹: 同调论
- [10] Alan Hatcher: Algebraic Topology
- [11] Edwin Spanier: Algebraic Topology
- [12] William Fulton: Algebraic Topology
- [13] M. Greenberg and J. Harper: Algebraic Topology: A First Course