Qualifying Exam 2025

August 29, 2025

All answers should be written in English. Clearly show your work.

Problem 1 (20 points)

Consider the following action for a system of a metric field $g_{\mu\nu}(x)$ and an electromagnetic field $A_\mu(x)$ in $3+1$ dimensions:

$$\int d^4x\sqrt{-g}\Big(R - \frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu}\Big).$$

(a) From the above action, derive the equation of motion for $A_\mu(x)$.

(b) From the above action, derive the stress tensor $T_{\mu\nu}$ for the electromagnetic field and show that it is conserved.

(c) Derive the Einstein equation for the system by considering the variation $\delta g^{\mu\nu}$ of the inverse metric. (You may use $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta R_{\mu\nu} = \nabla_\sigma \delta\Gamma^\sigma_{\mu\nu} - \nabla_\nu\delta\Gamma^\sigma_{\mu\sigma}$).

(d) Originally Einstein wrote an equation like $R_{\mu\nu} \propto T_{\mu\nu}$. Explain why this cannot be right. Also explain why the correct Einstein's equation makes sense.

Problem 2 (30 points)

Consider the Schwarzschild metric:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

(a) This is a black hole solution. Where is the horizon? How many Killing vectors does this solution have? Please write them down.

(b) Show that a massless particle moving with $\dot{\theta} = 0$ can be reduced to a one-dimensional problem with an effective potential. Write down the effective potential in terms of the energy E and angular momentum L and r, and plot it.

(c) What is the radius of the circular orbit? Is it stable?

Problem 3 (50 points)

Consider the ϕ^4 theory with Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

(Here we use the Minkowski metric with signature $(+,-,-,-)$.) Let us study the $2 \to 2$ scattering amplitude at the next-to-leading order $\mathcal{O}(\lambda^2)$.

(a) Draw all the one-particle irreducible (1PI) Feynman diagrams that contribute to the $2 \to 2$ scattering amplitude at order $\mathcal{O}(\lambda^2)$. For each diagram, indicate whether it is (i) divergent or finite, and (ii) dependent on the external momenta or a constant.

(b) Using the Feynman rules, write down the integral expression for the scattering amplitude $i\mathcal{M}(p_1 p_2 \to p_3 p_4)$ at order $\mathcal{O}(\lambda^2)$. Express your answer using the variables $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$ and $u(p_1-p_4)^2$, where p_1, p_2 are incoming external momenta and p_3, p_4 are outgoing external momenta.

(c) The divergent part of the amplitude is most clearly isolated using dimensional regularization ($d = 4-\epsilon$). The divergent piece is a constant, independent of the external momenta. Calculate this constant at order $\mathcal{O}(\lambda^2)$.

(d) The total $\mathcal{O}(\lambda^2)$ amplitude must be made finite through renormalization. This requires introducing a counterterm $\delta\mathcal{L}$ from the renormalization of the coupling constant λ. Write down the relevant counterterm Lagrangian $\delta\mathcal{L}_\lambda$ needed to cancel the divergences found in the 1-loop amplitude. Draw the corresponding Feynman diagram for this counterterm and write its contribution to $i\mathcal{M}$.

(e) After renormalization, the physical, finite coupling constant λ_{phys} is defined at a specific renormalization scale μ^2. Briefly explain the purpose of this scale and how it appears in the renormalized amplitude.

You may find the following equations useful:

$$(AB)^{-1} = \int_0^1 dx \; [xA + (1-x)B]^{-2} \,,$$

$$\int d^d k \, \frac{1}{(-k^2 - 2p\cdot k - M^2 + i\epsilon)^s} = (-1)^s \, i\pi^{d/2} \, \frac{\Gamma(s-d/2)}{\Gamma(s)} \, (-p^2 + M^2 - i\epsilon)^{d/2-s},$$

Here $\Gamma(z)$ is the Gamma function with a pole at $z=0$:

$$\Gamma(z) = \frac{1}{z} + \gamma + O(z),$$

with γ being the Euler constant.