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## Qualifying Exam-Analysis and Differential Equations

Spring 2026

1. (10 points) Compute the Fourier transform of the function  $u(x) = \frac{1}{1+x^4} \in L^1(\mathbb{R})$ , that is, for any  $\xi \in \mathbb{R}$ , compute the following integral

$$\hat{u}(\xi) = \int_{\mathbb{R}} \frac{e^{-ix\xi}}{1+x^4} dx.$$

2. (10 points) Find the solution to the initial value problem for  $u = u(x)$ ,

$$x \frac{d^2 u}{dx^2} - 2(x+1) \frac{du}{dx} + (x+2)u = 0, \quad u(1) = e, \quad u'(1) = 2e.$$

(Hint: First check that  $u = e^x$  is a solution to the equation.)

3. (10 points) Prove that the following limit exists in  $\mathcal{D}'(\mathbb{R})$  (in the sense of distribution on  $\mathbb{R}$ ), and find the limit:

$$\lim_{n \rightarrow +\infty} n^2 |x| \cos(nx).$$

4. (10 points) Let  $X, Y$  be Banach spaces. Let  $T : X \rightarrow Y$  be a bounded linear map. Suppose that  $T$  maps every bounded closed set to a closed set. Prove that  $T(X)$  is closed in  $Y$ .

5. (15 points) Let  $\Omega \subset \mathbb{C}$  be open and connected, and  $f_n : \Omega \rightarrow \mathbb{C}$  be holomorphic (i.e., analytic) for each  $n \in \mathbb{N}$ . Suppose  $f_n$  converges to  $f$  uniformly on each compact subset of  $\Omega$ , and  $f$  is non-constant. Let  $U \subset \Omega$  be open. For each compact subset  $K \subset f(U)$ , prove that  $K \subset f_n(U)$  for all sufficiently large  $n$ .

6. (15 points) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be locally integrable (i.e. Lebesgue integrable on any compact interval). Assume that  $f$  is weakly differentiable, that is, its weak derivative  $f'$  (in the sense of distribution) is also a locally integrable function. Prove that, except for a zero measure set,  $f$  agrees with a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  that is absolutely continuous on any compact interval.

7. (15 points) Given  $f \in C^1([0, \infty))$ , consider the linear differential equation for  $u = u(x)$  on  $[0, \infty)$

$$\frac{d^2u}{dx^2} - (1 + f(x))u = 0.$$

(1) (5 points) If  $\lim_{x \rightarrow +\infty} f(x) = 0$ , prove that any non-zero solution  $u = u(x)$  has at most finite number of zeroes on  $[0, \infty)$ ;

(2) (10 points) If  $\int_0^\infty |f(x)|dx < \infty$ , prove that there is a unique solution  $u = u(x)$  satisfying

$$\lim_{x \rightarrow +\infty} e^x u(x) = 1.$$

8. (15 points) Let  $\mathcal{H}$  be an infinite-dimensional separable complex Hilbert space. Consider the following two questions:

(1) (8 points) Let  $T$  be a bounded linear operator on  $\mathcal{H}$ . Suppose that for any orthonormal basis  $\{f_n\}_{n=1}^\infty$  in  $\mathcal{H}$  we have  $\lim_{n \rightarrow \infty} T f_n = 0$ . Prove that  $T$  is a compact operator. (Hint: Approximate  $T$  by finite-rank operators defined using a fixed orthonormal basis.)

(2) (7 points) Let  $\mathcal{H} = l^2(\mathbb{Z}_+)$ . Prove that there is a bounded linear operator  $T$  on  $l^2(\mathbb{Z}_+)$  such that

$$\begin{aligned} & T(a_1, a_2, a_3, \dots) \\ &= \left( a_1, \frac{1}{\sqrt{2}}(a_2 + a_3), \frac{1}{\sqrt{3}}(a_4 + a_5 + a_6), \frac{1}{\sqrt{4}}(a_7 + a_8 + a_9 + a_{10}), \dots \right) \end{aligned}$$

holds if  $\{a_n\}$  contains only finitely many non-zero terms, and that  $T$  is not a compact operator. (Hint: Consider the adjoint of  $T$ .)

Note: Part (2) shows that the condition “for every orthonormal basis” in (1) cannot be weakened to “for some orthonormal basis”.