清华大学考试试题专用纸

考试课程: _____ 姓名: ___________ 学号: ___________

Problems:

1. (10 pts) On a Riemannian manifold (M, g), suppose that f is a smooth function such that $|\operatorname{grad} f| = 1$. Show that the integral curves of $\operatorname{grad} f$ are geodesics.

2. (10 pts) Let M be the quotient space
$$([0,1] \times \mathbb{CP}^2)/(0, [z_0, z_1, z_2]) \sim (1, [\bar{z}_0, \bar{z}_1, \bar{z}_2]).$$
Compute the homology group $H_k(M; \mathbb{Z})$ for all $k \geq 0$.

3. (10 pts) Consider a map $f : \mathbb{T}^2 \to \mathbb{RP}^2$ from the torus to the projective plane. Suppose the induced map $f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{RP}^2)$ is trivial. Is f always null-homotopic? Prove your conclusion.

4. (10 pts) Let $F\colon (M, g) \to (M, g)$ be an isometry that fixes $p \in M$. Show that $dF_p = -\mathrm{id}$ on T_pM if and only if $F^2 = \mathrm{id}_M$ and p is an isolated fixed point.

5. (15 pts) Let M be a closed smooth manifold. Show that M is a smooth fiber bundle over S^1 if and only if there exists a closed, nowhere vanishing differential 1-form on M. (A differential form α is nowhere vanishing if for any $x \in M$, there exists $v \in T_xM$ such that $\alpha(v) \neq 0$.)

6. (15 pts) Consider an Einstein metric (N^{n-1}, g_N) with $\mathrm{Ric} = \frac{n-2}{n-1}\lambda g_N, \lambda < 0$. Find a
$$\rho\colon \mathbb{R} \to (0, \infty)$$
such that $(M^n, g) = (\mathbb{R} \times N, dr^2 + \rho^2(r) g_N)$ becomes an Einstein metric with $\mathrm{Ric} = \lambda g$.

7. (15 pts) (1) Let M be a compact, orientable $(2n+1)$-dimensional manifold with boundary N. Consider the map
$$i^* : H^n(M; \mathbb{R}) \to H^n(N; \mathbb{R})$$
induced by the inclusion $i : N \to M$. Show that
$$\text{dimension of the image of } i^* = \frac{1}{2}\text{dimension of } H^n(N; \mathbb{R}).$$
(2) Show that for any $n \geq 1$, $\#_n\mathbb{CP}^2$ is not the boundary of a compact, orientable 5-dimensional topological manifold. Here $\#_n\mathbb{CP}^2$ denotes the connected sum of n copies of complex projective space $\mathbb{CP}^2$.

8. (15 pts) Suppose (M, g) is a Riemannian manifold. There is a unique bundle endomorphism

$$\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$$

called the curvature operator of g, that satisfies

$$g(\mathcal{R}(W \wedge X), Y \wedge Z) = -R(W, X, Y, Z)$$

for all tangent vectors W, X, Y, Z at a point of M. A Riemannian metric is said to have positive curvature operator if $\mathcal{R}$ is positive definite.

(a) Show that positive curvature operator implies positive sectional curvature.

(b) Let g be the Fubini-Study metric on $\mathbb{CP}^2$. Do it have positive sectional curvature? Does g have positive curvature operator? Give details to justify your conclusions.