清华大学考试试题专用纸

考试课程 Subject: 求真书院代数博士生资格考 Algebra qualify exam for Qiuzhen Colledge

姓名 Name: ____________

学号 Student ID: ____________

- 考试时间: 2024 年 2 月 Exam Time: 2024.02
- 本试卷共 3 页, 9 道大题, 总分为 100 分. This exam has 3 pages, 9 problems, 100 points.
- 考生默认遵守考试纪律, 不遵守者后果自负. Candidates are expected to adhere to the examination discipline by default, and those who do not comply will bear the consequences themselves.
- 所有的解答请写出必要的细节, 推理依据和推理过程. All answers must include necessary details, reasoning basis, and the process of reasoning.

以下题目中 **Q** 是有理数域，**R** 指实数域, **C** 指复数域, **Z** 指整数全体, F_q 有 q 个元素的有限域. **在题目中环均指含幺交换环**.

In the following problems, **Q** is field of rational numbers, **R** is field of real numbers, **C** is field of complex numbers, **Z** is ring of integers, F_q is finite field consisting of q elements. **In all the problems, ring means commutative ring with multiplicative unit**.

题 1 (10 points)**.** *假设 V 是有限维复线性空间，给定 $T\colon V \to V$ 是线性变换. 证明*

1. *存在可对角化的线性变换 $D\colon V \to V$ 和幂零线性变换 $N\colon V \to V$，使得 $T = D + N$，并且 $DN = ND$.*
2. *这样的 D 和 N 被 T 唯一确定.*

Let V be a finite dimensional complex vector space. Given a linear transformation $T\colon V \to V$, prove

1. *There exists diagonalizable linear transformation $D\colon V \to V$ and nilpotent transformation $N\colon V \to V$, such that $T = D + N$, and $DN = ND$.*
2. *Such D and N are uniquely determined by T.*

题 2 (10 points)**.** *假设 A 和 B 是 n 阶正定实对称方阵，证明* $\det(A + B) > \det(A)$.

Let A and B be positive definite real symmetric matrices of order n，prove that $\det(A + B) > \det(A)$.

题 3 (10 points). 假设 G 为取定的 n 阶有限交换群. 求有多少不同的从 G 到 $\mathbf{R}/\mathbf{Z}$ 的群同态? 证明你的结论.

Let G be a fixed finite abelian group of order n. How many distinct group homomorphisms are there from G to $\mathbf{R}/\mathbf{Z}$? Prove your conclusion.

题 4 (10 points). 假设 D_{2n} 是阶数为 $4n$ 的二面体群. 写下 D_{2n} 的不可约复表示的特征标表.

Let D_{2n} be dihedral group of order $4n$. Write down the character table of irreducible complex representations of D_{2n}.

题 5 (10 points). 假设 p 是素数, 域 $F = F_p(x, y)$, $\alpha \in F$. 令 K 是 $x^p - x - \alpha$ 在 F 上的分裂域.

1. 证明 K/F 是 Galois 扩张.
2. 求所有可能的 Galois 群 $Gal(K/F)$.

Let p be a prime number and field $F = F_p(x, y)$, $\alpha \in F$. Assume K is the splitting field of $x^p - x - \alpha$ over F.

1. *Prove that K/F is Galois extension.*
2. *Find all the possible Galois group $Gal(K/F)$.*

题 6 (10 points). 假设 R 是整环, $\operatorname{Aut} R$ 是 R 的环自同构群. 对 $\operatorname{Aut} R$ 的有限子群 G, 定义

$$R^G = \{r \in R | g(r) = r, \forall g \in G\}.$$

1. 证明 R 在 R^G 上整.
2. 证明如果对 $\operatorname{Aut} R$ 的有限子群 G 和 H 有 $R^G = R^H$, 则 $G = H$.

Let R be an integral domain, $\operatorname{Aut} R$ the ring automorphism group of R. For finite subgroup G of $\operatorname{Aut} R$, define

$$R^G = \{r \in R | g(r) = r, \forall g \in G\}.$$

1. *Prove that R is integral over R^G.*
2. *Prove that for finite subgroups G and H of $\operatorname{Aut} R$, if $R^G = R^H$, then $G = H$.*

题 7 (10 points). 假设 A 和 B 都是域 k 上的有限生成代数, 且有环同态 $f\colon A \to B$ 使得 f 限制在 k 上是恒等映射. 对任意 B 中的极大理想 m, 证明 $f^{-1}(m)$ 是 A 的极大理想.

Let k be a field, A and B be finitely generated k-algebra, and $f\colon A \to B$ be a ring homomorphism such that the restriction of f on k is identity. Prove that for any maximal ideal m of B, the inverse image $f^{-1}(m)$ is a maximal ideal of A.

题 8 (15 points). 令 W 是 2×2 的迹零实矩阵组成的实线性空间, $W = \{A \in M_{2\times 2}(\mathbf{R}) | trace(A) = 0\}$.

1. 求 W 上的对称双线性型 $\langle A, A' \rangle = \operatorname{trace}(AA')$ 的符号

2. 证明 $P \star A = PAP^{-1}$ 定义了 $\mathrm{SL}(2, \mathbf{R})$ 在 W 上的线性作用.

3. 利用这个作用定义群同态 φ: $\mathrm{SL}(2, \mathbf{R}) \to O_{2,1}$. 其中 $O_{2,1}$ 是保持符号为 $(2, 1)$ 的实对称双线性的线性变换群.

4. 找到这个同态的核.

Let W be the real vector space of real trace-zero 2×2 matrices $W = \{A \in M_{2\times 2}(\mathbf{R}) | trace(A) = 0\}$.

1. *Find the signature of the symmetric bilinear form defined by $\langle A, A' \rangle = \mathrm{trace}\,(AA')$.*

2. *Prove that $P \star A = PAP^{-1}$ defines a linear group operation of $\mathrm{SL}(2, \mathbf{R})$ on the space W.*

3. *Use this operation to define a group homomorphism φ: $\mathrm{SL}(2, \mathbf{R}) \to O_{2,1}$. Here $O_{2,1}$ is the group of linear transformations preserving real symmetric bilinear form of signature $(2, 1)$.*

4. *Find the kernel of this homomorphism.*

题 9 (15 points)**.** 假设 p 是一个素数，$q = p^n$ 是素数幂. 考虑有限域 F_q 上的 2×2 的可逆矩阵组成的有限群 $\mathrm{GL}(2, F_q)$. 求

1. $\mathrm{GL}(2, F_q)$ 中的共轭类的个数.

2. 每个共轭类中的元素个数.

Let p be a prime number and $q = p^n$ a power of p. Consider the group $\mathrm{GL}(2, F_q)$ consisting of 2×2 invertible matrices over finite field F_q. Find

1. *The number of conjugacy classes in $\mathrm{GL}(2, F_q)$.*

2. *The number of elements in each conjugacy class.*