

PhD Qualification Exam in Applied Mathematics

Autumn, 2026

1. (20 points) Write $\|v\| = \sqrt{v^T v}$. Given a matrix $A \in \mathbb{R}^{n \times n}$, consider the Rayleigh quotient iteration:
 - 1: choose $q_0 \in \mathbb{R}^n$ with $\|q_0\| = 1$.
 - 2: $\mu_0 = q_0^T A q_0$, $k = 0$.
 - 3: **loop**
 - 4: $z_{k+1} = (A - \mu_k I)^{-1} q_k$.
 - 5: $q_{k+1} = \frac{z_{k+1}}{\|z_{k+1}\|}$.
 - 6: $\mu_{k+1} = q_{k+1}^T A q_{k+1}$.
 - 7: $k \leftarrow k + 1$.
 - 8: **end loop**

Let $Q_k = [q_k \ U_k]$ be an orthonormal matrix, and write $Q_k^T A Q_k = \begin{bmatrix} \mu_k & h_k^T \\ g_k & C_k \end{bmatrix}$. Also write $\rho_k = \|A q_k - \mu_k q_k\|$.

Suppose $\rho_k \rightarrow 0$ as $k \rightarrow \infty$. Prove:

- (a) $\rho_k = \|g_k\|$.
 - (b) $q_{k+1} = \delta_k Q_k \begin{bmatrix} 1 \\ y_k \end{bmatrix}$ where $y_k = (\mu_k I - C_k)^{-1} g_k$, $\delta_k = \frac{\text{sign}(h_k^T y_k)}{\sqrt{1 + y_k^T y_k}}$. Here $\text{sign}(x) \in \{\pm 1, 0\}$ is the sign of x .
 - (c) For sufficiently large k , there exists an orthonormal matrix W_k such that $g_{k+1} = -\delta_k W_k y_k y_k^T h_k + O(\rho_k^3)$.
 - (d) For sufficiently large k , $\rho_{k+1} = O(\rho_k^2)$.
 - (e) If A is symmetric, for sufficiently large k , $\rho_{k+1} = O(\rho_k^3)$.
2. (20 points) Consider the monic orthogonal polynomials $P_n(x)$ for the inner product $\langle f, g \rangle = \int_{-1}^1 \frac{f(x)g(x)}{1+x^2} dx$. For $f \in C[-1, 1]$, define $\|f\| = \sqrt{\langle f, f \rangle}$ and $\|f\|_\infty = \max_{x \in [-1, 1]} |f(x)|$.
 - (a) Compute $P_n(x)$ for $n = 0, 1, 2, 3$.
 - (b) For $f \in C^3[-1, 1]$, choose the interpolation nodes $x_0 \leq x_1 \leq x_2$ as the three zeros of $P_3(x)$, and denote its degree-two interpolation polynomial by L_2 . Find C_1 such that $\|f - L_2\| \leq C_1 \|f^{(3)}\|_\infty$ holds as sharply as possible.
 - (c) For $f \in C^6[-1, 1]$, derive the Gauss quadrature formula of $I(f) = \int_{-1}^1 \frac{f(x)}{1+x^2} dx$ for 3 quadrature nodes, written by $I_3(f)$. Find C_2 such that $|I(f) - I_3(f)| \leq C_2 \|f^{(6)}\|_\infty$ holds as sharply as possible.

3. (10 points) To solve ODE

$$\begin{cases} \frac{dy}{dx} = f(x, y), a \leq x \leq b, \\ y(a) = y_0, \end{cases}$$

consider the two-step scheme:

$$y_{n+2} - (1 + \alpha)y_{n+1} + \alpha y_n = \frac{h}{12} [(5 + \alpha)f(x_{n+2}, y_{n+2}) + 8(1 - \alpha)f(x_{n+1}, y_{n+1}) - (1 + 5\alpha)f(x_n, y_n)],$$

where $-1 \leq \alpha < 1$. Determine its interval of absolute stability.

4. (15 points) For the system

$$u_t = v_x, \quad v_t = u_x,$$

analyze the truncation error and stability of the scheme

$$\begin{aligned} \frac{1}{\tau} (u_j^{n+1} - u_j^n) &= \frac{1}{2h} (v_{j+1}^n - v_{j-1}^n), \\ \frac{1}{\tau} (v_j^{n+1} - v_j^n) &= \frac{1}{2h} (u_{j+1}^{n+1} - u_{j-1}^{n+1}). \end{aligned}$$

5. (15 points) For the parabolic equation

$$u_t = (a(x, t)u_x)_x, \quad a(x, t) \geq a_0 > 0, \quad (x, t) \in (0, l) \times (0, T); \quad u(x, 0) = \phi(x),$$

subject to periodic boundary conditions, construct a finite difference scheme that is second-order accurate in both space and time, and analyze its stability under suitable assumptions.

You can choose either 6 or 7 to answer. The points will be decided as $\max(6, 7)$.

6. (20 points) Solve the next two problems.

- (a) (6 points) Find the leading term of $\int_0^{\pi/2} (t^2 + 2)e^{i\lambda \cos t} dt$ as $\lambda \rightarrow +\infty$ and give the order of the remainder. Rigorous estimate is not required.

- (b) (14 points) For $0 < \epsilon \ll 1$, using the matched asymptotic expansion to construct the leading order asymptotic solution, which is uniformly valid for $t \in [0, 1]$, of the following boundary value problem for $y = y(t)$,

$$\epsilon y'' + 2y' + e^y = 0, \quad y(0) = y(1) = 0.$$

7. (20 points) Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a proper, closed, and μ -strongly convex function, where $\mu > 0$.

- (a) (6 points) Prove that f has a unique global minimizer.
(b) (7 points) Let

$$h(x) = \alpha \|x\|_1 + \beta \|x\|_2, \quad \alpha, \beta > 0.$$

For arbitrary $t > 0$ and $v \in \mathbb{R}^n$, compute the proximal operator of h given by

$$\text{prox}_{th}(v) = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \alpha \|x\|_1 + \beta \|x\|_2 + \frac{1}{2t} \|x - v\|_2^2 \right\}.$$

Give an explicit formula and justify your derivation.

- (c) (7 points) Independently, let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex function whose gradient is L -Lipschitz continuous, i.e.,

$$\|\nabla g(x) - \nabla g(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y \in \mathbb{R}^n.$$

Prove that ∇g is $1/L$ -cocoercive; that is, for all $x, y \in \mathbb{R}^n$,

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq \frac{1}{L} \|\nabla g(x) - \nabla g(y)\|_2^2.$$