# Qiuzhen College PhD Entrance Exam

Name: ____________ Student ID: ____________

- Exam Time: 2025 Spring
- This exam consists of 3 pages, 6 problems. The full score is 100 points.
- Examinees are expected to adhere to the examination discipline by default. Those who fail to abide by it shall bear the consequences themselves.
- All solutions should include necessary details, the basis of reasoning, and the reasoning process. When citing a theorem or conclusion, it is advisable to cite its original version as much as possible instead of an uncommon variant. If the question requires proving a theorem or conclusion itself, one cannot simply state its name without providing a proof.

In the following problems, $\mathbf{Q}$ is the field of rational numbers, $\mathbf{R}$ is the field of real numbers, $\mathbf{C}$ is the field of complex numbers, $\mathbf{Z}$ is the ring of integers. For any matrix $A$, we denote by $A^T$ the transpose of $A$.

1. (10 points) Let $V = M_n(\mathbf{R})$ be the vector space of $n \times n$ matrices over $\mathbf{R}$.

   (a) Let $W_1$ be the subspace of $V$ consisting of symmetric matrices, and $W_2$ the subspace of $V$ consisting of skew-symmetric matrices. Show that $V = W_1 \oplus W_2$.

   (b) Let $\mathcal{B}$ be the bilinear form on $V$ defined by $\mathcal{B}(A, B) = \mathrm{tr}(AB)$, where tr is the trace of a matrix. Determine the signature $(p, q)$ of $\mathcal{B}$ (i.e.: $p$ and $q$ are, respectively, the number of positive and negative elements in the diagonalization of $\mathcal{B}$).

2. (10 points) Let $\alpha$ and $\beta$ be two nonzero column vectors in $\mathbf{R}^n$ and $A$ be an $n \times n$ matrix defined by $A = \alpha\beta^T$. The standard inner product on $\mathbf{R}^n$ is defined by $\langle x, y\rangle = x^T y$ for $x, y \in \mathbf{R}^n$.

   (a) If $\langle \alpha, \beta\rangle = 0$, is $A$ always diagonalizable over $\mathbf{R}$?

   (b) If $\langle \alpha, \beta\rangle \neq 0$, is $A$ always diagonalizable over $\mathbf{R}$?

   Prove your statements.

3. (15 points) Suppose $f : [a, b] \to \mathbf{R}$ is a smooth function.

   (a) Take $n$ distinct points $\xi_1, \cdots, \xi_n$ on $[a, b]$. Prove that there exists a unique polynomial $L$ with degree no larger than $n - 1$, such that for any $x \in [a, b]$ there exists $\xi \in [a, b]$ satisfying
$$f(x) - L(x) = \frac{\prod_{i=1}^{n}(x - \xi_i)}{n!} f^{(n)}(\xi)\,.$$

(b) Now set $\xi_i = \frac{b+a}{2} + \frac{b-a}{2}\theta_i$ for some $\theta_i \in [-1, 1]\,, i = 1, 2, \cdots, n$. Prove that the polynomial $L$ above satisfies

$$\int_a^b L(x)dx = \frac{b-a}{2} \cdot \sum_{i=1}^{n} c_i f(\xi_i)\,, \quad c_i = \int_{-1}^{1} \prod_{j \neq i} \frac{t - \theta_j}{\theta_i - \theta_j} dt\,.$$

4. (5 points) Compute the integral

$$f(x) = \int_{-\infty}^{+\infty} \frac{\cos x}{x^4 + 1} dx\,.$$

5. (30 points)

Consider a one-dimensional quantum mechanical system of a particle of mass $m$ moving along the $x$-axis in a harmonic oscillator potential $m\omega^2 x^2/2$, where $\omega$ is a positive real number. Neglect relativistic effects.

(a) Express the Hamiltonian in terms of the dimensionless variable

$$\eta = \sqrt{\frac{m\omega}{\hbar}}x. \tag{1}$$

and the ladder operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\eta + \frac{d}{d\eta}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\eta - \frac{d}{d\eta}\right). \tag{2}$$

Furthermore, derive the commutation relation between $\hat{a}$ and $\hat{a}^\dagger$.

(b) Express the position operator $\hat{x}$ and the momentum operator $\hat{p}$ using the ladder operators.

(c) Let $|0\rangle$ be the normalized ground state of the harmonic oscillator. Show that a normalized state $|n\rangle = c_n(\hat{a}^\dagger)^n|0\rangle$ is an eigenstate of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ with eigenvalue $n$, and determine the (positive real) normalization factor $c_n$.

(d) The variance of an operator is the expectation value of the operator squared minus the expectation value squared of the operator. Calculate the variances of $\hat{x}$ and $\hat{p}$ in the state $|n\rangle$ and derive an uncertainty relation.

(e) Consider the eigenvalue equation for the annihilation operator:

$$\hat{a}|\Phi\rangle = R|\Phi\rangle. \tag{3}$$

For simplicity, assume $R$ is real. Calculate the variances of $\hat{x}$ and $\hat{p}$ in the state $|\Phi\rangle$, and compare the result with the uncertainty relation obtained above.

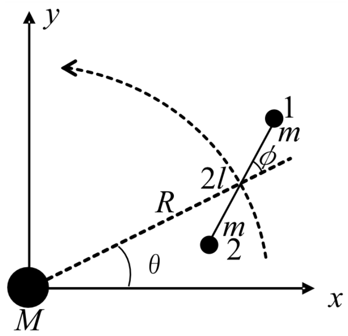
(f) Using the explicit form of $\hat{a}$, solve the differential equation to find the wavefunction $\langle x|\Phi\rangle$.

(g) In the Heisenberg picture, find the time dependence of $\hat{a}$ and $\hat{a}^\dagger$.

(h) Using the time dependence of $\hat{a}$ and $\hat{a}^\dagger$ obtained above and the expression of $\hat{x}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, calculate the expectation value $\langle\Phi|\hat{x}(t)|\Phi\rangle$ of the position operator $\hat{x}(t)$ in the Heisenberg picture. Furthermore, discuss the time dependence of the variance of $\hat{x}(t)$ in the state $|\Phi\rangle$.

6. (30 points)

Two point particles of mass $m$ are connected by a massless rod of length $2l$ and move in a plane, as shown in the picture below. A point particle of mass $M$ ("Earth") is fixed at the origin and exerts gravitational force (with Gravitational constant $G$) on this two-particle system.




(a) Take coordinates as shown in the picture. The $xy$-coordinates of the center of the rod is $(R\cos\theta, R\sin\theta)$. Relative to the center of the rod, particle 1 is located at $(l\cos(\theta+\phi), l\sin(\theta+\phi))$ and particle 2 is located at $(-l\cos(\theta+\phi), -l\sin(\theta+\phi))$. Express the kinetic energy of the system in terms of $(R, \theta, \phi)$ and their time derivatives.

(b) Express the distances $r_1$, $r_2$ to particles 1 and 2 in terms of $R$, $l$, $\phi$, and write down the Lagrangian of the two-particle system.

(c) Consider the limit $l \to 0$. Derive the Euler–Lagrange equation with respect to $R$. If this two-particle system is orbiting around Earth at a constant radius $R = R_0$, then its angular velocity $\dot{\theta} = \omega_0$ is a constant. Find $\omega_0$.

(d) Suppose $l \ll R$ and consider the situation where we can approximate $R = R_0$ (constant) and $\dot{\theta} = \omega_0$ (constant). Write down the Euler–Lagrange equation with respect to $\phi$, expand it using the approximation $(1+x)^{-\alpha} \approx 1 - \alpha x$ for $|x| \ll 1$. Finally, show that if $\phi$ is small, then $\phi$ undergoes a simple harmonic oscillation around $\phi = 0$. What is the frequency of this oscillation?