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Qualifying Exam-Analysis and Differential Equations

Fall 2026

1. (10 points) Find all equilibrium points for the system of differential equations for $(x, y) = (x(t), y(t))$,

$$x' = x(1 - x^2 - 6y^2), \quad y' = y(1 - 3x^2 - y^2).$$

and determine their stability in the sense of Lyapunov (asymptotically stable, stable but not asymptotically stable, unstable).

2. (10 points) Is there a one-to-one holomorphic function which maps $\{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $\{z \in \mathbb{C} : r < |z| < R\}$ where $R > r > 0$? Justify your answer.

3. (10 points) Let $s > 0$, and define the operator D^α as follows

$$\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi).$$

Show that

$$\|D^s(\varphi u)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{|\alpha| \leq s+1} \|\partial_x^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)} \|u\|_{H^s}, \quad \forall u, \varphi \in C_c^\infty(\mathbb{R}^n),$$

where C is a universal constant independent of u and φ .

4. (15 points) Let $(X, \mathcal{F}, \mu)$ be a σ -finite measure space and $0 < r < p < +\infty$.

- (a) (10 points) Show that there exists $C = C(p, r, X) > 0$, such that for all measurable function f

$$\frac{1}{C} \|f\|_{L_w^p} \leq \|f\| \leq C \|f\|_{L_w^p},$$

Here

$$\|f\|_{L_w^p} := \sup_{\lambda > 0} \lambda [\mu(\{x \in X : |f(x)| > \lambda\})]^{-\frac{1}{p}},$$

$$\|f\| := \sup_{\mu(E) < +\infty} \mu(E)^{-\frac{1}{r} + \frac{1}{p}} \left(\int_E |f(x)|^r d\mu(x) \right)^{\frac{1}{r}}$$

- (b) (5 points) Determine whether the above inequality still holds true if $(X, \mathcal{F}, \mu)$ is not σ -finite.

5. (10 points) For all positive integer m , we denote by

$$\mathcal{Y}_m = \left\{ f \in C^\infty(\mathbb{R}) : \forall k \in \mathbb{N}, \sup_{y \in \mathbb{R}} (1 + |y|)^{m+k} |\partial_y^k f(y)| = C_k < +\infty \right\}.$$

Show that $\mathcal{H}f \in \mathcal{Y}_1$ if $f \in \mathcal{Y}_2$. Here $\mathcal{H}f$ is the Hilbert transform of f :

$$\mathcal{H}f(x) = \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy.$$

6. (15 points) For $a \in \mathbb{C}$, consider the differential equation for $u = u(x)$

$$xu'' + 2u' + (a - x)u = 0.$$

- (a) (5 points) Prove that for every $a \in \mathbb{C}$, there is a nontrivial solution $u \in C^\infty(\mathbb{R})$.
 (b) (10 points) Denote $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$. Suppose that there is a nontrivial solution $u \in \mathcal{S}'(\mathbb{R})$ (tempered distribution) such that both $u|_{\mathbb{R}_\pm}$ are smooth and both limits $\lim_{x \rightarrow 0^\pm} u(x)$ exist, show that a must be an even number. (Hint: For example, when $a = 0$, the Dirac δ -function is a solution with the desired properties. In general, consider its Fourier transform $\hat{u}(\xi)$ and the behaviour when $\xi \rightarrow \pm\infty$.)

7. (15 points) Let $\Omega \neq \mathbb{C}$ be a simply connected open subset of $\mathbb{C}$. Suppose $f : \Omega \rightarrow \Omega$ is holomorphic and f is not the identity function. Prove that f has at most one fixed point in Ω .

8. (15 points) Let $u \in C_c^\infty(\mathbb{R}^2)$ and $A > 100$. Show that

$$\int_{|x|<A} |u(x)|^2 dx \leq CA^2 \log A \int_{\mathbb{R}^2} (|u(x)|^2 e^{-|x|} + |\nabla u(x)|^2) dx.$$

where C is a universal constant independent of u and A .