QUALIFY EXAM FOR APPLIED MATHEMATICS

(1) (20 points) The n-th Chebyshev polynomial is defined by: $T_n(x) = \cos(n\cdot\arccos(x))$. Answer the following questions related to $T_n(x)$.

(i) (6 points) Prove that $T_n(x)$ is a polynomial of degree n.
(ii) (6 points) For any x with $0 < x < 1$, show that $T_n(2x-1) = T_{2n}(\sqrt{x})$.
(iii) (8 points) Prove that the system of Chebyshev polynomials $\{T_k : 0 \le k < n\}$ is orthogonal with respect to the discrete inner product $(u, v) = \sum_{k=1}^{n} u(x_k)v(x_k)$, where $\{x_k\}$ are the Chebyshev points $x_k = \cos(\frac{2k-1}{2n}\pi)$.

(2) (15 points) Let A be a symmetric positive definite matrix. Assume that the conjuage gradient method is applied to solve the linear system $Ax = b$, where x^* is the exact solution.

(i) (10 points) Prove the following error estimate:

$$||x_k - x^*||_A \le \left(\frac{\sqrt{\kappa_2(A)} - 1}{\sqrt{\kappa_2(A)} + 1} \right)^k ||x_0 - x^*||_A \ ,$$

where x_k is the solution obtained by the k-th iteration, $||\cdot||_A$ denotes the A-norm defined by $||x||_A = \sqrt{x^* A x}$, and $\kappa_2(A) = \frac{\max \lambda(A)}{\min \lambda(A)}$ denotes the condition number of A under the l_2 norm. (If you apply any theorem from approximation theory, please state the theorem clearly. No need to prove that theorem.)
(ii) (5 points) Describe one Krylov subspace method for non-Hermitian matrices ($A \neq A^*$). (You can use pseudo-code or words/formulas as long as the steps are clear.)

(3) (15 points) Consider Kepler's equation

$$f(x) = x - \epsilon \sin x - \eta \ , (0 < |\epsilon| < 1) \ , \eta \in \mathbb{R} \ .$$

(i) (5 points) Show that for each ϵ, η, there is exactly one real root $\alpha = \alpha(\epsilon, \eta)$. Furthermore, $\eta - |\epsilon| \le \alpha \le \eta + |\epsilon|$.
(ii) (5 points) Writing the equation in fixed point form:

$$x = \varphi(x) \ , \quad \varphi(x) = \epsilon \sin x + \eta \ ,$$

show that the fixed point iteration $x_{n+1} = \varphi(x_n)$ converges for arbitrary staring value x_0.

(iii) (5 points) Let m be an integer such that $m\pi < \eta < (m+1)\pi$. Show that Newton's method with starting value

$$x_0 = \begin{cases} (m+1)\pi & \text{if } (-1)^m \epsilon > 0\ , \\ m\pi & \text{otherwise,} \end{cases}$$

is guaranteed to converge (monotonically) to $\alpha(\epsilon, \eta)$.

(4) (15 points) For the system

$$u_t = v_x, \quad v_t = u_x,$$

analyze the truncation error and stability of the scheme

$$\frac{1}{\tau}\left(u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)\right) = \frac{1}{2h}(v_{j+1}^n - v_{j-1}^n),$$
$$\frac{1}{\tau}\left(v_j^{n+1} - \frac{1}{2}(v_{j+1}^n + v_{j-1}^n)\right) = \frac{1}{2h}(u_{j+1}^n - u_{j-1}^n).$$

(5) (15 points) Write and prove the maximum principle of the centered finite difference scheme for discretizing the equation

$$u_{xx} + u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = 0, \quad f < 0,$$

under some suitable assumptions.

Remark: You can choose either (6) or (7). The points will be decided as $\max((6),(7))$**.**

(6) (20 points) Consider the following boundary value problem for $y = y(x)$ on $[0,1]$ as $0 < \epsilon \ll 1$,

$$\epsilon y'' + \epsilon(1+x)^2 y' - y = x - 1,\ y(0) = \alpha,\ y(1) = -1.$$

(i) (10 points) Suppose $\alpha = 1$. Construct a composite expansion of the above problem and sketch the solution.
(ii) (5 points) Construct a composite expansion of the above problem for $\alpha = 0$.
(iii) (5 points) What is the accuracy of your solution in ϵ? Formally explain your conclusion. Consider the first case only.

(7) (20 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and define $\partial f(x)$ to be the subgradient set of f at x.

(i) (10 points) If $x \in \textbf{int dom} f$, prove that $\partial f(x)$ is nonempty and bounded.
(ii) (5 points) Assume $f = \|x\|_1 = \sum_{i=1}^{n} |x_i|$, write down the $\partial f(x)$.
(iii) (5 points) Given $y \in \mathbb{R}^n$ and $\lambda > 0$, calculate the closed form solution of the problem:

$$\min_x \frac{1}{2}\|x - y\|_2^2 + \lambda \|x\|_1.$$