清华大学考试试题专用纸

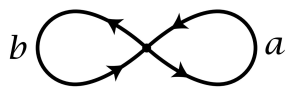
考试课程: _____　　姓名: ____________　　学号: ____________

表 1: 从以下 6 题中勾选出 4 题

5	6	7	8	9	10

必做题

1. (5 points+5 points) Let a, b denote the two standard generators of $\pi_1(X) \cong F_2$.



   (a) Draw a picture of the directed graph of the covering space of X that corresponds to the following subgroup

$$H = \langle a^2,\ b^2,\ ab^2a,\ ba^2b,\ (ab)^2 \rangle.$$

   On each edge of the directed graph, you should draw an arrow for its direction and label "a" or "b" for its image under the covering map.

   (b) Is the covering above a normal covering? Prove your claim.

2. (10 points) Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Prove that the complement $\mathbb{R}^n \setminus M$ is connected and simply-connected.

3. (10 pts) Let (M, g) be a complete Riemannian manifold. Is every Killing vector field X on M complete?

4. (10pts) Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be **self-dual** if $*\omega = \omega$, and **anti-self-dual** if $*\omega = -\omega$.

   (a) Show that every 2-form ω on M can be written uniquely as a sum of a self-dual form and an anti-self-dual form.

   (b) On $M = \mathbb{R}^4$ with the Euclidean metric,determine the self-dual and anti-self-dual forms in standard coordinates.

从以下 6 题中选做 4 题

5. (15 pts) Classify all 2024-dimensional complete Riemannian manifolds with constant sectional curvature $K = 1$.

6. (15 points) Let M be a 2-dimensional manifold without boundary such that $\pi_1(M)$ is an infinite group. (M is **not** necessarily compact.) Compute $\pi_n(M)$ for all $n \geq 2$. Prove your claim.

7. (15 pts) Compute the Ricci curvature and scalar of $\left(\mathbb{R}^2, g = e^{-\pi(x^2+y^2)}(dx \otimes dx + dy \otimes dy)\right)$. Is $\left(\mathbb{R}^2, g = e^{-\pi(x^2+y^2)}(dx \otimes dx + dy \otimes dy)\right)$ a complete Riemannian manifold? Prove your claim.

8. (10 points +5 points) Let M, N be closed, connected, orientable manifolds of the same dimension n. Suppose that $f : M \to N$ induces a nonzero map

$$f_* : H_n(M; \mathbb{Z}) \to H_n(N; \mathbb{Z}).$$

Are the following statements true? If true, give a proof; if false, give a counter-example and prove that the example you give is indeed a counter-example.

   (a) $\dim_{\mathbb{Q}} H_k(M; \mathbb{Q}) \geq \dim_{\mathbb{Q}} H_k(N; \mathbb{Q})$ for all k.

   (b) $\text{rank} H_k(M; \mathbb{Z}) \geq \text{rank} H_k(N; \mathbb{Z})$ for all k. (Here, the rank of a group is defined to be the minimal number of generators. By this definition, $\text{rank}(\mathbb{Z} \oplus \mathbb{Z}/2) = 2$.)

9. (15 pts) Can $M = \mathbb{S}^1 \times \mathbb{S}^2$ admit a smooth Riemannian metric g with $\text{Ric}(g) = f \cdot g$ for some smooth funtion f on M? Prove your claim.

10. (15 points) Prove that $\mathbb{RP}^n \times \mathbb{S}^2$ does not have an open cover consisting of k contractible open subsets if $k \leq n + 1$.