

General Relativity and Quantum Field Theory

Total time: 3 hours

Conventions: metric signature $(-, +, +, +)$ throughout; $G = c = 1$ unless stated; λ is an affine parameter; $\dot{} = d/d\lambda$. State any additional assumptions you make. Partial credit is given for setting up the problem correctly even if the final computation is not completed.

Part I — General Relativity (50 pts)

Problem 1 — The Poincaré-type NHEK geometry in u -coordinates (25 pts)

The *near-horizon geometry of an extreme Kerr black hole* (“NHEK”) is obtained as the smooth $\varepsilon \rightarrow 0$ near-horizon limit of extreme Kerr ($a = M$), a solution of the vacuum Einstein equation. In Poincaré-type coordinates (t, u, θ, φ) with $\varphi \sim \varphi + 2\pi$, $0 \leq \theta \leq \pi$, the metric is

$$ds^2 = 2J\Omega^2(\theta) \left[-u^2 dt^2 + \frac{du^2}{u^2} + d\theta^2 + \Lambda^2(\theta)(d\varphi + u dt)^2 \right], \quad (1)$$

with

$$\Omega^2(\theta) = \frac{1 + \cos^2 \theta}{2}, \quad \Lambda^2(\theta) = \frac{4 \sin^2 \theta}{(1 + \cos^2 \theta)^2}, \quad J = M^2. \quad (2)$$

The original black hole horizon is now at $u \rightarrow 0$, and the boundary $u \rightarrow \infty$ of NHEK matches to exterior Kerr.

(a) Vacuum solution?

- (i) From the information provided, can you tell whether the NHEK metric is a solution to the Einstein equation? Why?
- (ii) As a direct check, compute $R_{\theta\theta}$.

(b) Show that

$$\xi_1 = t \partial_t - u \partial_u, \quad \xi_2 = \left(t^2 + \frac{1}{u^2} \right) \partial_t - 2tu \partial_u - \frac{2}{u} \partial_\varphi$$

are Killing vectors of the metric. Are there any other Killing vectors? Compute the Lie brackets and comment on the algebra generated by the vectors.

- (c) Is there a globally timelike Killing vector — i.e. one that is timelike at every point of the NHEK patch ($u > 0$, $0 \leq \theta \leq \pi$, all t, φ)? If not, indicate where any candidate vector becomes spacelike or null.
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Problem 2 — Basic equatorial timelike geodesics in NHEK (25 pts)

- (a) Write the geodesic Lagrangian $\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. Show that $\omega \equiv -p_t$ and $m \equiv p_\varphi$ (with $p_\mu \equiv \partial\mathcal{L}/\partial\dot{x}^\mu$) are constants.
- (b) Prove that geodesics with $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$ remain at $\theta = \pi/2$ for all λ . For the following questions, restrict to the equatorial plane $\theta = \pi/2$.
- (c) Consider timelike geodesics (λ the proper time, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$) on the equatorial plane. Prove that the radial equation can be written as

$$\dot{u}^2 = V(u, \omega, m).$$

Find the expression of $V(u, \omega, m)$.

- (d) Sketch $V(u, \omega, m)$ qualitatively as a function of u . Classify the orbits: find the ranges of m for which the particle (i) is *bound* — oscillating in u , never reaching the boundary — and (ii) *escapes* to the boundary $u \rightarrow \infty$ (only asymptotically). Treat the marginal case and the case $\omega = 0$ separately. Can the particle reach the horizon $u \rightarrow 0$ in finite proper time? Is there any value of m for which circular orbits ($u = \text{const}$ with $0 < u < \infty$) exist?
- (e) Show that the radial equation can be written in the form

$$\ddot{u} = -\Omega^2(u - u_c),$$

where Ω^2 and u_c are both constants (Ω^2 may be negative). Determine Ω^2 and u_c . Solve the equation.

Part II — Quantum Field Theory (50 pts)

Conventions for Part II: metric signature $(-, +, +, +)$; $\hbar = c = 1$; dimensional regularization is performed in $d = 4 - 2\varepsilon$ dimensions; the $\overline{\text{MS}}$ pole subtraction uses $1/\bar{\varepsilon} \equiv 1/\varepsilon - \gamma_E + \ln 4\pi$. *Note: ε in this Part is the dimensional-regularization parameter and is unrelated to the near-horizon limit parameter of Part I.*

Problem 3 — Commutator and Feynman propagator (20 pts)

A free real scalar field of mass m .

(a) Compute the commutator $[\phi(x), \phi(y)]$ from the mode expansion. Evaluate it at equal times $x^0 = y^0$, compute $\partial_{x^0}[\phi(x), \phi(y)]|_{x^0=y^0}$, and show that $[\phi(x), \phi(y)]$ itself vanishes for all spacelike separations $(x - y)^2 < 0$.

(b) Compute the Feynman propagator

$$D_F(x - y) = \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle$$

in position space. Calculate its Fourier transform, and compute $(\partial^2 + m^2)D_F(x - y)$.

Problem 4 — One-loop ϕ^4 : renormalization and unitarity (30 pts)

A single real scalar with $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$, in $d = 4 - 2\varepsilon$ dimensions.

(a) Compute the one-loop self-energy $\Sigma(p^2)$ in dimensional regularization, defined by

$$\langle T \phi(p) \phi(-p) \rangle_{\text{conn}} = D_F(p) + D_F(p) [-i\Sigma(p^2)] D_F(p) + \cdots,$$

and give $\Sigma(p^2)$ explicitly through $O(\varepsilon^0)$.

(b) Compute the one-loop bubble

$$B(P^2) = -i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\varepsilon)((k + P)^2 - m^2 + i\varepsilon)}$$

using Feynman parameters and dimensional regularization, and write down the one-loop four-point amplitude $i\mathcal{M}(s, t, u)$.

(c) Work in the $\overline{\text{MS}}$ -bar scheme: subtract only the $1/\bar{\varepsilon}$ pole terms, where $\frac{1}{\bar{\varepsilon}} \equiv \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi$, keeping no finite parts. Compute the counterterms δm^2 and $\delta \lambda$ that render the theory finite through one loop, and state the relation between bare and renormalized parameters.

(d) Compute the one-loop beta function $\beta(\lambda) = \mu d\lambda/d\mu$ and its value at $d = 4$. What does its sign imply for the UV behavior of the coupling?

(e) Compute $\text{Im } \mathcal{M}(s, t, u)$ for $s > 4m^2$ at order λ^2 from your result in (b), and verify the optical theorem for the s -channel bubble at this order.