

Select three out of four problems to work out. You must declare which three you choose, otherwise the first three will be graded.

Classical Mechanics

A double pendulum consists of one pendulum attached to another. Consider a double pendulum with masses m_1 and m_2 attached by rigid massless wires of length $l_1 = l_2 = l$. Let ϕ_1, ϕ_2 denote the angles that each wire makes with the vertical, as shown in the figure. Let g denote the acceleration due to gravity. The motion is in the plane shown and frictionless.

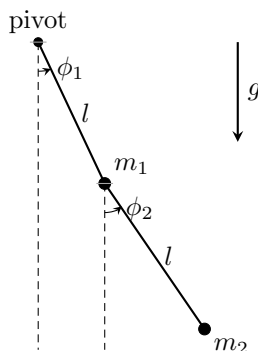


Figure 1: Double pendulum with equal-length rigid massless wires and angular coordinates ϕ_1, ϕ_2 .

1. Derive the Lagrangian. Express your result in terms of l, m_1, m_2, g and ϕ_1, ϕ_2 and their derivatives.
2. Determine the generalized momenta p_1, p_2 conjugate to ϕ_1, ϕ_2 .
3. Determine the Euler-Lagrange equations of motion of ϕ_1 and ϕ_2 .
4. Assume small oscillations, $|\phi_1|, |\phi_2| \ll 1$. Determine the characteristic frequencies.
5. What are the limiting values of the frequencies in the case $m_1 \gg m_2$? Interpret your result physically.

Electrodynamics

A point charge $(-Q)$ is surrounded by a *thin* non-conducting sphere of radius R with evenly distributed charge Q so that the entire system is neutral. The sphere has total mass M and can freely rotate around the center without friction. Initially, there is uniform external magnetic field B along the $\hat{z}$ axis, and the sphere is static.

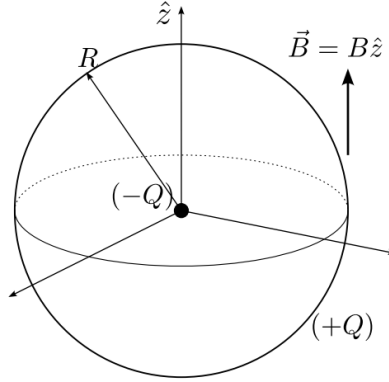


Figure 2: Charged sphere and external magnetic field configuration for the Electrodynamics problem.

1. Calculate the Poynting vector $\mathbf{S}$ at every point in space and the total angular momentum carried by the electromagnetic fields. *Hint: Spherical coordinates and basis $(\hat{r}, \hat{\theta}, \hat{\phi})$ are the most convenient choice.*
2. The external magnetic field is gradually reduced to zero. Calculate the final angular velocity ω of the sphere. Neglect the magnetic field created by the sphere itself. *Hint: The moment of inertia of a thin sphere is $I = \frac{2}{3}MR^2$.*
3. Compare the result of previous two questions and provide a qualitative explanation.
4. Calculate the magnetic field inside and outside of the charged sphere rotating with angular velocity $\omega\hat{z}$. One potential approach is:
 - (a) calculate the magnetic field B_1 in the center of the rotating sphere.
 - (b) calculate the total magnetic dipole moment m of the rotating sphere.
 - (c) assuming that the magnetic field outside the sphere is that of a *point-like magnetic dipole* m , calculate the magnetic field $B_2(r, \theta, \phi)$ for $r > R$;
 - (d) show that the magnetic field equal to B_2 *just outside* the sphere and B_1 *just inside* the sphere satisfies the Maxwell equations.

Alternative methods to calculate the magnetic field will also be accepted.

Quantum Mechanics

In a highly simplified model of a molecule, an electron moves in the vicinity of three nuclei which include one “A” nucleus and two “B” nuclei, inter-linked as

shown in the figure. In absence of any coupling between electron states bound to different nuclei, the electron has energy $E_1^{(0)}$ when it is on the “A” nucleus and energy $E_2^{(0)}$ when it is on a “B” nucleus, such that $E_2^{(0)} - E_1^{(0)} = \Delta > 0$. The state of the electron can be defined by its location, labeled as (1)-(3) in the figure: $|1\rangle = (100)$, $|2\rangle = (010)$, $|3\rangle = (001)$.

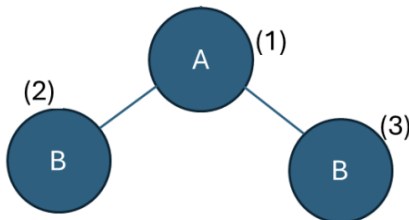


Figure 3: Simplified three-nucleus molecular model for the Quantum Mechanics section.

1. Under the approximation that the electron cannot move from one nuclei to another, write down the Hamiltonian in the basis of state $|1\rangle$, $|2\rangle$, and $|3\rangle$.
2. Now we will allow the electron to move between an A atom and a B atom with a coupling energy of a , but not between the B atoms. Assume $a \ll \Delta$. Write down the perturbation Hamiltonian associated with A-B coupling.
3. Calculate perturbation to energy $E_1^{(0)}$ to the second order and calculate the 1st order perturbation to the energy $E_2^{(0)}$.
4. Now calculate the precise (without using perturbation method) eigenstates of the system and their energies. Compare your result with the perturbation calculation under the limit $a \ll \Delta$.
5. Assuming an electron at the vicinity of nuclei A at $t = 0$, what is the probability of finding the electron at the vicinity of a nuclei B at time t ?

Statistical Mechanics

Consider a one-dimensional chain of molecules consisting of N molecules which exist in two configurations α and β with corresponding energies ϵ_α and ϵ_β and lengths a and b . The chain is in thermal equilibrium at temperature τ and subject to a tensile force f (see the figure).

1. Find the partition function. *Hint: Don't forget to include the work done by the tensile force which contributes to the energy of the system.*

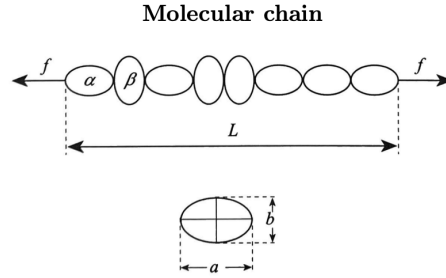


Figure 4: One-dimensional molecular chain under tensile force for the Statistical Mechanics problem.

2. Find the average length $\langle L \rangle$ of the chain as a function of f and temperature.
3. Assume that $\epsilon_\alpha > \epsilon_\beta$ and $a > b$. Estimate the average length $\langle L \rangle$ in the absence of the tensile force as a function of temperature. What are the high and low temperature limits? What do these correspond to physically? Sketch a plot of $\langle L \rangle$ versus τ .
4. Calculate the linear response function (Hooke's constant) of the chain

$$\chi = \left(\frac{\partial \langle L \rangle}{\partial f} \right)_{f=0}. \quad (1)$$

and show that it is greater than zero. Why should this be the case?