Qualifying Exam: 2025 Fall

考试课程: Probability & Statistics　　姓名: ____________　　学号: ____________

- There are 11 problems in this exam (3 pages). You need to choose 8 of them to solve. If you select more than 8, only the first 8 that you have worked on will be graded. Note that 4 of the problems are worth 15 points each and the rest 10 points each.
- You must follow all the rules of exam taking. Misconducts will be subject to proper disciplinary actions by the Center.
- You must provide all necessary details for full credits. A final answer with no or little explanation/derivation, even if correct, receives a minimal credit.
- $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of positive integers. $\xrightarrow{(d)}$ and $\overset{(d)}{=}$ mean "converges in distribution" and "equal in distribution", respectively.
- Beta function $Beta(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ where $\Gamma(\cdot)$ is the gamma function. $\Gamma(n) = (n-1)!$ for any positive integer n.
- For $\mu \in \mathbb{R}, \sigma > 0$, the 1-dimensional normal distribution $\mathcal{N}(\mu, \sigma^2)$ is defined by the density function
$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

1. (10 points) Let X and N be two independent random variables. Suppose X follows the standard exponential distribution (i.e., with probability density function $e^{-x}\mathbf{1}_{\{x>0\}}$), and N follows the Poisson distribution with parameter $\lambda > 0$.

   (a) Find all $\lambda > 0$ such that $\mathbb{E}[X^N] < \infty$.

   (b) Calculate the expectation of $\mathbb{E}[X^N]$ under the above condition on λ.

   (c) Is there any $\lambda > 0$ such that $\mathrm{Var}[X^N] < \infty$? Justify your answer.

2. (10 points) Let X, Y be two independent random variables uniformly distributed on $[0, 1]$, and define
$$Z := \max\{X, Y\}.$$

   (a) Determine the distribution of Z conditioned on $\{X + Y \in [0, 1]\}$.

   (b) Determine the distribution of Z conditioned on $\{X + Y \in [1, 2]\}$.

3. (10 points) Let $(X_n, \mathcal{F}_n)$ be a martingale.

(a) Prove that if $(X_n^2, \mathcal{F}_n)$ is also a martingale, then $X_m = X_n$ a.s. for any $m, n \in \mathbb{N}$.

(b) Prove that if $(|X_n|^p, \mathcal{F}_n)$ is a martingale for some $p > 1$, then $(|X_n|^q, \mathcal{F}_n)$ is also a martingale for every $1 \le q \le p$.

4. (15 points) Given independent random variables $(X_n)_{n\in\mathbb{N}}$ satisfying

$$\forall n \in \mathbb{N}, \qquad \mathbb{P}(X_n = -1) = \frac{1}{2}, \qquad \mathbb{P}(X_n = 4n) = \frac{1}{8n}, \qquad \mathbb{P}(X_n = 0) = \frac{1}{2} - \frac{1}{8n},$$

and denote $S_n := \sum_{k=1}^n X_k$. Prove that almost surely

$$\liminf_{n\to\infty} \frac{S_n}{n} < 0 < \limsup_{n\to\infty} \frac{S_n}{n}.$$

5. (10 points) Let B_t be a one-dimensional (1D) standard Brownian motion starting from 0. Define $X_t = \exp(B_t - \frac{1}{2}t)$.

(a) Prove that X_t converges almost surely at $t \to \infty$.

(b) For any $t > 0$, calculate

$$\mathbb{E}\int_0^t X_s \mathrm{d}s, \quad \text{and} \quad \mathrm{Var}\int_0^t X_s \mathrm{d}s.$$

(c) Show that

$$\int_0^\infty X_t \mathrm{d}t < \infty \quad a.s.$$

6. (15 points) Let M be a 2×2 real symmetric random matrix of the form

$$M = \begin{pmatrix} X_1 & Y \\ Y & X_2 \end{pmatrix},$$

where X_1, X_2, and Y are independent normal random variables: X_1 and X_2 follow the normal distribution $\mathcal{N}(0,2)$ with mean 0 and variance 2, and Y follows the standard normal distribution $\mathcal{N}(0,1)$ with mean 0 and variance 1. Let $\lambda_1 \ge \lambda_2$ denote the eigenvalues of M, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding unit eigenvectors.

(a) Find the distribution of $\lambda_1 + \lambda_2$.

(b) Find the distribution of $\lambda_1 - \lambda_2$.

(c) Are λ_1 and λ_2 independent? Prove your claim.

(d) Prove that the joint distribution of $(\mathbf{v}_1, \mathbf{v}_2)$ is invariant under orthogonal transformations, i.e., for any 2×2 orthogonal matrix O, $(O\mathbf{v}_1, O\mathbf{v}_2)$ has the same distribution as $(\mathbf{v}_1, \mathbf{v}_2)$.

7. (10 points) Suppose $\hat{\theta}_n$ is a real-valued consistent estimator of a parameter of interest θ based on n observations, and it is further known that $\theta \in [-1, 1]$. Let T_n be a sufficient statistic for θ. Show there exists a consistent estimator of θ based on T_n.

8. (15 points) Let $X_1, ..., X_n$ be iid $\sim \exp(\lambda)$ (with pdf $\lambda e^{-\lambda x}$ for $x > 0$). Let $\phi = P_\lambda(X_1 > x) = e^{-\lambda x}$.

   (a) Find a complete and sufficient statistic for λ.

   (b) Find the UMVUE of ϕ.

9. (15 points) Let $X_1, ..., X_n$ be iid Bernoulli(p) random variables with $0 < p < 1$.

   (a) Let $g(p) = p^k + (1-p)^{n-k}$ where k is a nonnegative integer and $0 \le k \le n$. Find the BUE (UMVUE) for $g(p)$.

   (b) Prove that $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ is an admissible estimator/rule of p under the loss function $l(p, a) = \frac{(p-a)^2}{p(1-p)}$.

10. (10 points) Let $X_1, ..., X_n$ be iid from the uniform distribution $\mathrm{U}(0, \theta)$ with $\theta > 0$ unknown. Suppose that the prior on θ is lognormal, i.e., $\ln \theta$ follows $\mathrm{N}(\mu_0, \sigma_0^2)$ distribution where $\mu_0 \in \mathbb{R}$ and $\sigma_0 > 0$ are known constants.

    (a) Find the posterior density of $\ln \theta$.

    (b) Suppose that one defines the Bayes estimate for θ as the value that maximizes the posterior density of θ. Find this Bayes estimator. Is it consistent for θ? Please explain your answer.

11. (10 points) Let $X_1, ..., X_n$ be iid random variables following a discrete uniform distribution on the set $1, 2, ..., \theta$, where θ is an unknown positive integer. Let θ_0 be a known positive integer. Let $X_{(n)}$ be the largest order statistic and $x_{(n)}$ be its realized value. We are interested in the hypothesis testing problems below at level $0 < \alpha < 1$.

    (a) Consider testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$. Show that

    $$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } x_{(n)} > \theta_0 \\ \alpha & \text{if } x_{(n)} \le \theta_0 \end{cases}$$

    is a (randomized) UMP test of size α.

    (b) Now consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$. Show that

    $$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } x_{(n)} > \theta_0 \text{ or } x_{(n)} \le \theta_0 \alpha^{1/n} \\ 0 & \text{otherwise} \end{cases}$$

    is a UMP test of size α when $\theta_0 \alpha^{1/n}$ is an integer.