清华大学考试试题专用纸

考试课程 Subject: 求真书院代数博士生资格考 Algebra qualify exam for Qiuzhen College

姓名 Name: ____________

学号 Student ID: ____________

- 考试时间: 2024 年 9 月 Exam Time: 2024.09
- 本试卷共 3 页, 9 道大题, 总分为 100 分. This exam has 3 pages, 9 problems, 100 points.
- 考生默认遵守考试纪律, 不遵守者后果自负. Candidates are expected to adhere to the examination discipline by default, and those who do not comply will bear the consequences themselves.
- 所有的解答请写出必要的细节, 推理依据和推理过程. All answers must include necessary details, reasoning basis, and the process of reasoning.

以下题目中 $\mathbf{Q}$ 指有理数域, $\mathbf{R}$ 指实数域, $\mathbf{C}$ 指复数域, $\mathbf{Z}$ 指整数全体, $\mathbf{F}_q$ 是有 q 个元素的有限域. **在题目中环均指含幺交换环**.

In the following problems, $\mathbf{Q}$ is the field of rational numbers, $\mathbf{R}$ is the field of real numbers, $\mathbf{C}$ is the field of complex numbers, $\mathbf{Z}$ is the ring of integers, $\mathbf{F}_q$ is the finite field consisting of q elements. **In all the problems, ring means commutative ring with multiplicative unit**.

题 1 (10 points)**.** 设 q 是一个素数的幂. 计算 $\mathbf{F}_q$ 上次数为 21 的首一不可约多项式的数量.

Suppose q is a prime power. Compute the number of monic irreducible polynomials of degree 21 over $\mathbf{F}_q$.

题 2 (10 points)**.** 给定正整数 n. 考虑所有迹为零的 $n+1$ 阶复方阵构成的李代数 $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbf{C})$. 令 $\mathfrak{h}$ 为 $\mathfrak{g}$ 中的对角矩阵构成的子空间.

(i) 写出 $\mathfrak{g}$ 关于 $\mathfrak{h}$ 的 *Cartan* 分解（也称作根分解）.

(ii) 描述 $(\mathfrak{g}, \mathfrak{h})$ 的根系中的所有根，并证明你的结论.

Given a positive integer n. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbf{C})$ be the Lie algebra of all $(n+1)\times(n+1)$ complex matrices with trace 0. Let $\mathfrak{h}$ be the space of all diagonal matrices in $\mathfrak{g}$.

(i) Give the Cartan decomposition (also called root decomposition) of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

(ii) Describe all roots in the root system associated with $(\mathfrak{g}, \mathfrak{h})$, and prove your result.

题 3 (10 points)**.** 给定素数 p. 考虑群

$$G = \{ \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \big| a \in \mathbf{F}_p, b \in \mathbf{F}_p^{\times} \} \subset \mathrm{GL}(2, \mathbf{F}_p).$$

其中群运算由矩阵乘法给出. 求出 G 的不可约复表示的特征表.

Let p be a prime number. Consider group

$$G = \{ \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \big| a \in \mathbf{F}_p, b \in \mathbf{F}_p^{\times} \} \subset \mathrm{GL}(2, \mathbf{F}_p)$$

with group law given by matrix multiplication. Find the character table for irreducible complex representations of G*.*

题 4 (10 points)**.** 设有限非交换单群 G 满足 $|G| \leq 60$. 求所有可能的 G.

Suppose G *is a finite non-abelian simple group such that* $|G| \leq 60$*. Find all possible* G*.*

题 5 (12 points)**.** 设 q 是一个素数的幂.

(i) 证明任何 $\mathbf{F}_q$ 上的幂零方阵都与一个约当标准形矩阵相似.

(ii) 计算 $\mathbf{F}_q$ 上所有 3×3 幂零矩阵的数量.

Suppose q is a prime power.

(i) Prove that any nilpotent square matrix over $\mathbf{F}_q$ *is similar to a matrix of the Jordan normal form.*

(ii) Compute the number of 3×3 *nilpotent matrices over* $\mathbf{F}_q$*.*

题 6 (12 points)**.** 给定交换环 R, 考虑如下 R 模同态构成的交换图表

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \quad (1)$$

满足

(i) 所有列都是短正合列;

(ii) 第一行与第三行是短正合列;

(ii) 第二行 $A_2 \to B_2 \to C_2$ 的复合是零同态.

证明第二行也是一个短正合列.

Let R be any commutative ring, and consider the commutative diagram (1) *(see above) of R-module homomorphisms. Suppose we have*

(i) *All the columns are short exact sequences;*

(ii) *The first and third rows are short exact sequences;*

(ii) *The composition $A_2 \to B_2 \to C_2$ is zero.*

Prove that the second row is a short exact sequence.

题 7 (12 points)**.** 令 $\mathbf{F} = \mathbf{Q}[\cos \frac{\pi}{15}]$.

(i) 证明 $\mathbf{F}/\mathbf{Q}$ 为伽罗瓦扩张.

(ii) 求 $\mathrm{Gal}(\mathbf{F}/\mathbf{Q})$.

(iii) 求一个域扩张链 $\mathbf{Q} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_k = \mathbf{F}$, 使得 $[\mathbf{F}_{i+1} : \mathbf{F}_i]$ 为素数.

Let $\mathbf{F} = \mathbf{Q}[\cos \frac{\pi}{15}]$.

(i) *Prove $\mathbf{F}/\mathbf{Q}$ is a Galois extension.*

(ii) *Find the structure of* $\mathrm{Gal}(\mathbf{F}/\mathbf{Q})$.

(iii) *Find a chain $\mathbf{Q} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_k = \mathbf{F}$ of field extensions, such that $[\mathbf{F}_{i+1} : \mathbf{F}_i]$ are prime numbers.*

题 8 (12 points)**.** 考虑环 $\mathbf{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$.

(i) 证明 $\mathbf{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ 是一个欧几里得整环.

(ii) 设 $p \geq 5$ 是一个素数. 设 $\alpha \in \overline{\mathbf{F}}_p \backslash \mathbf{F}_p$ 是一个元素, 使得 $\alpha^3 = -1$. 推导 $(\alpha - \alpha^{-1})^2 = -3$, 并证明 p 在 $\mathbf{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ 中不可约当且仅当 $p \equiv 2$ *(mod 3)*.

Consider the ring $\mathbf{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$.

(i) *Prove that $\mathbf{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is a Euclidean domain.*

(ii) *Let $p \geq 5$ be a prime number. Let $\alpha \in \overline{\mathbf{F}}_p \backslash \mathbf{F}_p$ be an element so that $\alpha^3 = -1$. Deduce that $(\alpha - \alpha^{-1})^2 = -3$, and show that p is irreducible in $\mathbf{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ if and only if $p \equiv 2$ (mod 3).*

题 9 (12 points)**.** 给定素数 p. 设 R 是一个局部环, 满足对任意元素 $r \in R$ 均有 $r^p = r$. 令 I 为 R 的极大理想.

(i) 求 R/I 的所有可能性;

(ii) 求 R 的所有可能性.

Let p be a prime number. Suppose R is a local ring such that for any element $r \in R$, we have $r^p = r$. Let I be the maximal ideal of R.

(i) Find all the possibilities of R/I;

(ii) Find all the possibilities of R.