PhD Qualification Exam in Applied Mathematics

Autumn, 2025

1. (10 points) Except LU-type factorizations, matrix inversion can be obtained by iterations. Given $A \in \mathbb{R}^{n\times n}$, consider the sequence $\{X_k\}$ generated by:

$$X_{k+1} = 2X_k - X_k A X_k, \quad k = 0, 1, 2, \ldots.$$

Prove:

(a) if $\|I - AX_0\| < 1$, then A is nonsingular and $X_k \to A^{-1}$ quadratically.

(b) there exists X_0 such that $\|I - AX_0\| < 1$, if and only if A is nonsingular.

2. (20 points) For any continuous function f, define $\|f\|_\infty = \max_{x\in[-1,1]} |f(x)|$.

Define the n-th Chebyshev polynomial by: $T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

(a) Show that $T_n(x) = \begin{cases} \cos(n \arccos x), & |x| \le 1, \\ \cosh(n \operatorname{arccosh} x), & |x| > 1. \end{cases}$

(b) Prove that $\min_{\substack{\text{polynomial } p \\ \deg p \le n, p(1)=1}} \|p\|_\infty = 1$, which is attained at $p = T_n$.

(c) For $f \in C^{n+1}[-1, 1]$, choose the interpolation nodes $x_0, x_1, \ldots, x_n$ as the $n+1$ zeros of $T_{n+1}(x)$, and denote its degree-n interpolation polynomial by L_n. Prove

$$\|f - L_n\|_\infty \le \frac{\|f^{(n+1)}\|_\infty}{(n + 1)!2^n}.$$

(d) For $f \in C^{2n+2}[-1, 1]$, derive the Gauss quadrature formula of $I(f) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}}\, \mathrm{d}x$ for $n+1$ quadrature nodes, written by $I_{n+1}(f)$, and prove

$$|I(f) - I_{n+1}(f)| \le \frac{\pi\|f^{(2n+2)}\|_\infty}{(2n + 2)!2^{2n+1}}.$$

3. (20 points) In many applications generalized eigenvalue problem is needed to be solved: finding $x \in \mathbb{C}^n$ and $\mu, \lambda \in \mathbb{C}$, s.t. $x \ne 0, |\mu|^2 + |\lambda|^2 = 1, \mu Ax = \lambda Bx$ for given $A, B \in \mathbb{C}^{n\times n}$. Here x and (λ, μ) are called the eigenvector and eigenvalue of the pair (A, B) respectively.

If B is nonsingular, this problem is equivalent to the standard eigenvalue problem $AB^{-1}y = \mu^{-1}\lambda y$.

For two unitary matrices Q, Z, $Q^H AZ = S, Q^H BZ = T$ is called a UET performing on (A, B), written as $Q^H (A, B)Z = (S, T)$ for ease.

(a) Give an example that (A, B) has more than n eigenvalues.

(b) Prove that there exists a UET such that $Q^H (A, B)Z = (S, T)$, where S, T are upper triangular.

(c) Can you obtain the eigenvalues of (A, B) from (S, T)? How?

(d) Propose a method to construct a UET such that $Q_0^H (A, B)Z_0 = (A_0, B_0)$, where A_0 is upper Hessenberg, and B_0 is upper triangular. For ease, you may assume $n = 4$.

4. (15 points) Construct the Du Fort-Frankel scheme for the diffusion equation in 2D $u_t = u_{xx} + u_{yy}$ and discuss its consistency and stability.

5. (15 points) For the wave equation $u_{tt} = u_{xx}$, analyze the stability of the scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{4h^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2h^2} + \frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{4h^2}.$$

You can choose either 6 or 7 to answer. The points will be decided as $\max(6, 7)$**.**

6. (20 points) Consider the so-called Rayleigh oscillator

$$y'' + y + \epsilon[\frac{1}{3}(y')^3 - y'] = 0,$$

with initial condition

$$y(0) = 0, \quad y'(0) = 2a.$$

where $y = y(t), \quad \epsilon > 0, \quad a > 0$

(a) (14 points) For a small ϵ, construct an approximation of the solution to the above problem which is valid for large t.

(b) (4 points) What is the accuracy of this approxmation? State a conclusion and briefly explain it.

(c) (2 points) Plot the approximated orbits in the phase plane, i.e., $y - y'$ plane for several different a. And explain what you observe.

7. (20 points) Consider the minimization

$$\min_{x \in \mathbf{R}^n} \|Ax - b\|_2 + \gamma\|x\|_1,$$

with $A \in \mathbf{R}^{m\times n}, b \in \mathbf{R}^m$, and $\gamma > 0$.

(a) (8 points) Derive the Lagrange dual of the equivalent problem:

$$\min_{x,y} \|y\|_2 + \gamma\|x\|_1, \text{ s.t. } Ax - b = y,$$

with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

(b) (8 points) Suppose $Ax^* - b \neq 0$ where x^* is an optimal point. Define $r = \dfrac{Ax^* - b}{\|Ax^* - b\|_2}$. Show that

$$\left\|A^\top r\right\|_\infty \leq \gamma, \quad r^\top Ax^* + \gamma \left\|x^*\right\|_1 = 0.$$

(c) (4 points) Show that if $\|a_i\|_2 < \gamma$ where a_i is the i-th column of A, then $x_i^* = 0$.