Instruction for choosing courses in the direction Algebra and Number Theory

Beginner courses:

- Algebra 0
- Algebra 1 or Algebra 1 (H)
- Algebra 2 or Algebra 2 (H)

Two preparatory courses:

- Representation Theory
- Commutative Algebra (listed as a graduate course)

Both are offered in Spring, once every year.

Nine core courses:

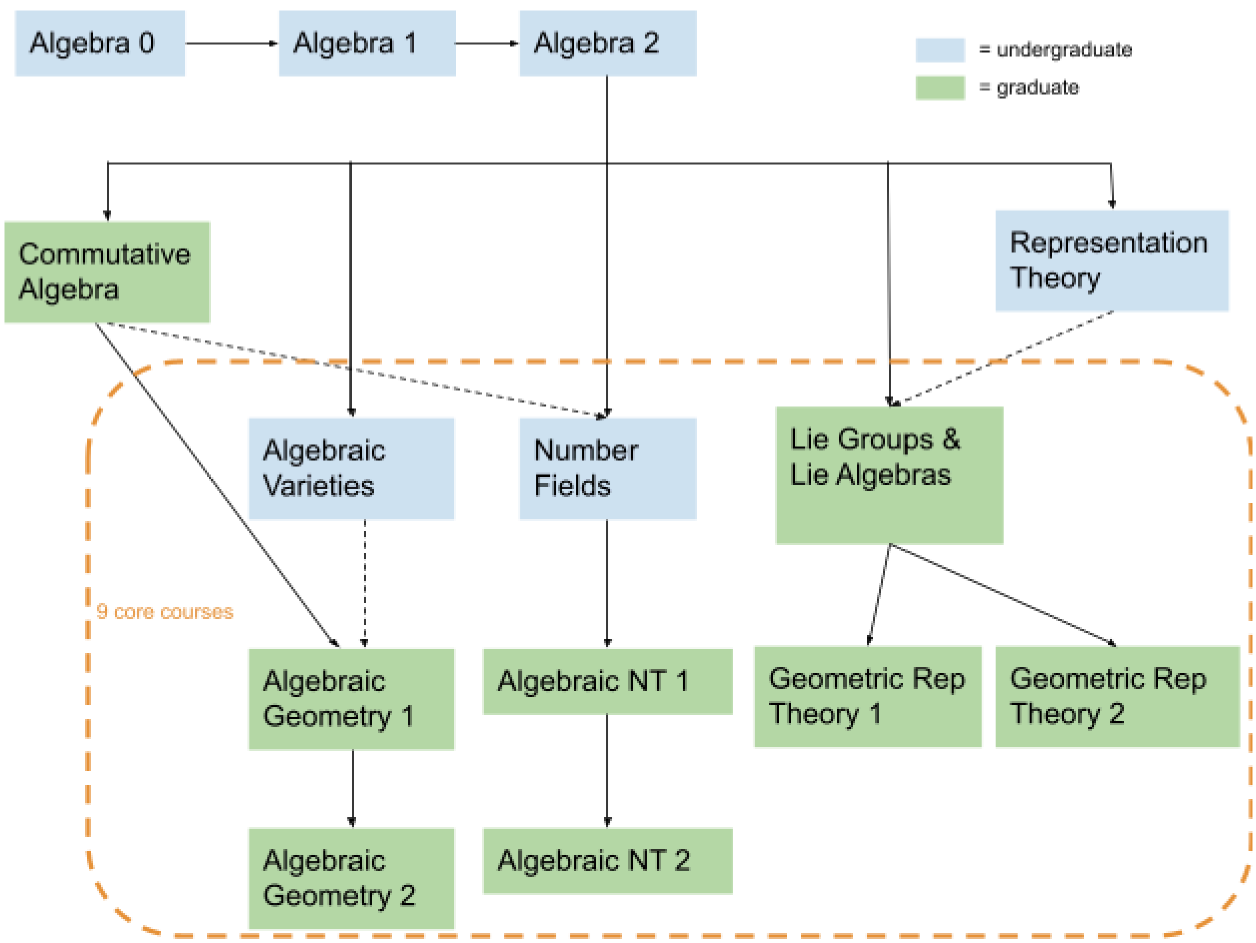
Grouped into three series. **Each will be offered once per year.** The courses in each series form the minimum foundation to work in the corresponding research direction. Typically the students are expected to take these courses before/while finding a PhD advisor.

Algebraic Geometry Series ↓	Algebraic Number Theory Series ↓	Representation Theory Series ↓
Algebraic Varieties (S/F)	Number Fields (F)	Lie Groups and Lie Algebras (F)
Algebraic Geometry 1 (Intro to schemes) (F)	Algebraic Number Theory 1 (Statement of Class Field Theory and applications) (S)	Geometric Representation Theory 1 (Kazhdan-Lusztig Theory) (S)
Algebraic Geometry 2 (Cohomology) (S)	Algebraic Number Theory 2 (Proof of CFT) (F)	Geometric Representation Theory 2 (Springer Theory) (F)

F=Fall, S=Spring   [blue] = undergraduate   [green] = graduate

Prerequisites

- Algebra 2 is needed for all the more advanced courses.
- Commutative Algebra is needed for Algebraic Geometry 1 and strongly recommended for Number Fields.
- Representation Theory is recommended for Lie Groups and Lie Algebras, but not strictly necessary.
- Algebraic Varieties is strongly recommended for Algebraic Geometry 1, but not strictly necessary.
- Geometric Representation Theory 2 is quite independent from Geometric Representation Theory 1, and does NOT assume the latter as a prerequisite.



Course pattern (starting Spring 2025)

<table>
<tr><th></th><th>Beginner course</th><th>Prep course</th><th>AG series</th><th>ANT series</th><th>RT series</th></tr>
<tr><td rowspan="2">Spring 2025</td><td>Algebra 0</td><td>Representation Theory</td><td>Algebraic Varieties</td><td rowspan="2">Algebraic Number Theory 1</td><td rowspan="2">Geometric Rep Theory 1</td></tr>
<tr><td>Algebra 2</td><td>Commutative Algebra</td><td>Algebraic Geometry 2</td></tr>
<tr><td rowspan="2">Fall 2025</td><td rowspan="2">Algebra 1</td><td rowspan="2"></td><td>(Algebraic Varieties, here or Spring 2026)</td><td>Number Fields</td><td>Lie Groups & Lie Algebras</td></tr>
<tr><td>Algebraic Geometry 1</td><td>Algebraic Number Theory 2</td><td>Geometric Rep Theory 2</td></tr>
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Syllabus of the two preparatory courses

Representation theory

Prerequisite: Algebra 1, Algebra 2

Textbook:
P. Etingof et al, Introduction to Representation Theory

Reference:
Jean-Pierre Serre, Linear Representations of Finite Groups

Contents

Chapter 1. Basic Notions
1.1 Algebras, Representations, Ideals
1.2 Quivers, Lie algebras, Tensor product
1.3 Representations of sl(2)
1.4 Subrepresentations, filtrations
1.5 Characters of representation
1.6 Jordan-Holder theorem

Chapter 2. Representations of finite groups
2.1 Maschke's theorem
2.2 Dual and tensor representations
2.3 Orthogonality of characters
2.4 Character tables
2.5 Burnside's theorem
2.6 Induced representations, Frobenius reciprocity
2.7 Mackey formula
2.8 Representations of Sn

Chapter 3. Representations of GL(V)
3.1 Schur Weyl duality
3.2 Schur polynomial, Weyl character formula
3.3 Representations of GL_2(F_q)
3.4 Root system, Weyl groups, semisimple Lie algebra

Chapter 4. Quiver representations
4.1 Indecomposable representations
4.2 Root system
4.3 Gabriel's theorem
4.4 Reflection functors

Commutative Algebra

Preliminary: Algebra-1, Algebra-2

References:

M. Atiyah & I. MacDonald, Introduction to commutative algebra
D. Eisenbud, Commutative algebra, with a view toward algebraic geometry

Contents

1. Rings
a) Basic notions, ring homomorphisms
b) Prime ideals, maximal ideals
c) Extension and contraction of ideals
d) Spectrum of rings
2. Modules
a) Homomorphism of modules, submodules and quotient modules
b) Direct sum of modules, finitely generated modules
c) Exact sequence of modules, tensor product and flat modules
3. Localization
a) Basic definition and universal property
b) Extension and contraction of ideals by localization
4. Primary decomposition
a) Primary ideals
b) Uniqueness of primary decomposition
5. Integrality
a) Integral ring extensions
b) Going up and going down

c) Integral closure and valuation ring

6. Noetherian rings and Artin rings

a) Ascending chain condition and descending chain condition

b) Primary decomposition of Noetherian rings

c) Structure theorem of Artin rings

7. Dedekind domains

a) Discrete valuation and discrete valuation rings

b) Dedekind domains and fractional ideals

8. Completion

a) Topologies and completions

b) Graded rings and graded modules

c) Hensel's lemma

9. Dimension theory

a) Hilbert functions

b) Dimension theory of Noetherian local rings

c) Regular local rings

Syllabus of the nine core courses

Algebraic Varieties

Preliminary: Algebra 1, 2

Textbook:
Eduard Looijenga, Algebraic varieties, Surveys of Modern mathematics 15, Higher Education Press

References:
Joe Harris: Introduction to Algebraic Geometry
R. Hartshorne, Algebraic geometry, Springer-Verlag 1997.
Gortz and Wedhorn, Algebraic Geometry I and II, Springer-Verlag 2023.

This course will discuss algebraic varieties over an algebraically closed field.

Contents

1. Affine varieties
   1.1 Spec construction, Zariski topology, irreducibility
   1.2 Hilbert's Nullstellensatz
   1.3 Category of affine varieties
   1.4 Sheaf of regular functions, notion of prevarieties and varieties
   1.5 Product of varieties
   1.6 Function fields and rational maps
   1.7 Finite morphisms

2. Local properties of varieties
   2.1 Dimension
   2.2 Smooth and singular points
   2.3 Differentials and derivations
   2.4 Sheaves on varieties

3. Projective varieties
   3.1 Projective spaces and its Zariski topology
   3.2 The Proj construction and blowing up
   3.3 Proper morphisms
   3.4 Segre and Veronese embeddings, Grassmannians and Plucker embeddings
   3.5 Hilbert polynomials and Bezout theorem

4. Algebraic curves
  4.1 Valuations and points
  4.2 Divisors and invertible sheaves
  4.3 Riemann-Roch theorem for curves
  4.4 Residues and Serre duality
  4.5 Riemann-Hurwitz theorem

5. Algebraic surfaces (If time permits)
  5.1 Intersection theory
  5.2 Riemann-Roch theorem for surfaces
  5.3 Hodge index theorem

Algebraic Geometry 1

Preliminary: Algebra 1, 2, Commutative algebra.

Textbook:
Lei Fu, Algebraic geometry, Higher Education Press.

References:
R. Hartshorne, Algebraic geometry, Springer-Verlag 1997.
Gortz and Wedhorn, Algebraic Geometry I and II to the reference, Springer-Verlag 2023.

Contents
1. Presheaves and sheaves
  1.1 Definitions and examples of presheaves and sheaves
  1.2 Abelian categories
  1.3 Sheaf associated to presheaves
  1.4 Direct images and inverse images of sheaves, adjoint functors

2. Schemes
  2.1 Spectra of commutative rings
  2.2 locally ringed spaces, schemes
  2.3 Proj S

3. Properties of schemes and morphisms
  3.1 Connected schemes, irreducible schemes, irreducible components, noetherian spaces, reduced schemes, integral schemes, quasi-compact morphisms
  3.2 Noetherian schemes, morphisms of finite types, affine morphism.
  3.3 Open immersion, closed immersions, immersions, reduced induced closed subschemes, gluing of schemes

3.4 Fibered product, separated morphisms, proper morphisms, projective morphisms.

4. Coherent sheaves
   4.1 Quasi-coherent sheaves and coherent sheaves on schemes
   4.2 Quasi-coherent sheaves and coherent sheaves on schemes on Spec A, their direct images and inverse images under affine morphisms, tensor product, Hom.
   4.3 Quasi-coherent sheaves and coherent sheaves on schemes on Proj A, Serre twisting sheaves, Serre's theorem
   4.4 Kahler differentials

Algebraic Geometry 2

Preliminary: Algebra 1, 2, Commutative algebra, Algebraic geometry 1

Textbook:
Lei Fu, Algebraic geometry, Higher Education Press.

References:
R. Hartshorne, Algebraic geometry, Springer-Verlag 1997.
Gortz and Wedhorn, Algebraic Geometry I and II to the reference, Springer-Verlag 2023.

Contents
1. Derived functors on abelian categories
   1.1 Injective resolutions and projective resolutions
   1.2 Derived functors, universal cohomological functors
   1.3 Cohomology, higher direct images, Ext, Tor
   1.4 Flasque sheaves, Godement resolutions

2. Spectral sequences
   2.1 Definition and general properties of spectral sequences
   2.2 The spectral sequence for a filtered complex
   2.3 The spectral sequence for a bicomplex
   2.4 Cartan-Eilenberg resolution, Grothendieck spectral sequences, Leray spectral sequences, other important spectral sequences

3. Cech cohomology and calculation of cohomology
   3.1 Cech cohomology, alternating Cech complex, Cech cohomology as the cohomology theory on the abelian category of presheaves
   3.2 Spectral sequence relating Cech cohomology and cohomology of sheaves, Cartan's lemma
   3.3 Koszul cohomology, cohomology of coherent sheaves on affine schemes and on Proj S

3.4 Finiteness of cohomology for projective morphisms, Serre's vanishing theorem

4. Applications of cohomology theory
   4.1 Finiteness of cohomology for proper morphisms
   4.2 Completion of cohomology, theorem on formal functions
   4.3 Theorem of connectedness
   4.4 Riemann-Roch theorem
   4.5 Stein factorization, Zariski Main Theorem (optional)
   4.6 Serre duality (optional)
   4.6 Semicontinuity theorem, cohomological base change theorem (optional)

Number Fields

Preliminary: Algebra 1, 2.

References:

D. A. Marcus, Number Fields (2nd edition), Springer, 2018
A. Fröhlich and M. J. Taylor, Algebraic Number Theory, Cambridge University Press, 1991
S. Lang, Algebraic number theory (2nd edition), Springer, 1994
J. Neukirch, Algebraic Number Theory, Springer, 1999
Z. I. Borevich and I. R. Shafarevich, Number Fields, Academic Press, 1986
J-P. Serre, A course in arithmetic, Springer, 1978

Contents

1. Dedekind domains
   1.1 Algebraic integers
   1.2 Discrete valuation rings and Dedekind domains
   1.3 Fractional ideals

2. Extensions
   2.1 Decomposition and ramification
   2.2 Discriminant and different
   2.3 Frobenius automorphisms

3. Class groups and units

3.1 Lattices and Mikowski's theory
3.2 Ideal class group
3.3 Dirichlet's unit theorem

4. Local fields
4.1 Valuations and completions
4.2 Nonarchimedean local fields
4.3 Local-global principle

Algebraic Number Theory 1

Preliminary: Algebra 1, 2, Number Fields.

References:
D. Cox, Primes of the Form x^2+ny^2: Fermat, Class Field Theory, and Complex Multiplication (2nd edition), Wiley, 2013
D. Ramakrishnan and R. J. Valenza, Fourier Analysis on Number Fields, Springer, 1999
J.-P. Serre, Local Fields, Springer, 1980
J.W.S. Cassels and A. Fröhlich, Algebraic Number Theory (2nd edition), London Mathematical Society, 2010
E. Artin and J. Tate, Class Field Theory, AMS Chelsea Publishing, 2008

Contents
1. Adeles and ideles
1.1 Topological groups, rings, and fields
1.2 Review of global fields and local fields
1.3 Adeles and ideles
1.4 Approximation and finiteness theorems

2. Class field theory
2.1 Statements of local class field theory
2.2 Statements of global class field theory in terms of ideles
2.3 Statements of global class field theory in terms of ideals

3. Applications of class field theory

3.1 The Chebotarev density theorem
3.2 Artin L-functions

4. Advanced topics (at least one of the following topics)

4.a Tate’s thesis
4.b Galois representations
4.c Arithmetic groups and reduction theory (genus theory of quadratic forms)
4.d Other classical applications

Note: the course ``Number Fields’’ may not cover Dedekind zeta functions and Dirichlet L-functions.

Algebraic Number Theory 2

Preliminary: Algebra 1, 2, Number Fields, Number Theory 1.

References:

J.S. Milne, Class Field Theory, available at https://www.jmilne.org/math/CourseNotes/cft.html
J.-P. Serre, Local Fields, Springer, 1980
J.W.S. Cassels and A. Fröhlich, Algebraic Number Theory (2nd edition), London Mathematical Society, 2010
E. Artin and J. Tate, Class Field Theory, AMS Chelsea Publishing, 2008
S. Lang, Algebraic number theory (2nd edition), Springer, 1994
J. Neukirch, Algebraic Number Theory, Springer, 1999

Contents

1. Lubin-Tate theory

1.1 Lubin-Tate formal group laws
1.2 Local Kronecker-Weber theorem

2. Group cohomology

2.1 Definition and properties of group cohomology
2.2 Tate cohomology
2.3 Galois cohomology

3. Local class field theory
3.1 Galois cohomology of non-archimedean fields
3.2 Local Artin map
3.3 Proof of local class field theory

4. Global class field theory
4.1 Cohomology of ideles
4.2 First and second inequalities
4.3 Proof of global class field theory

Lie groups and Lie algebras

Preliminary: Algebra 1, 2.

Textbook:
W. Fulton and J. Harris, Representation theory, a first course, Springer-Verlag 2004.

References:
T. Brocker and T. Tom Dieck, Representations of compact Lie groups, Springer-Verlag 1985.
Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag 1980.

Contents
1. Basics
  1.1 Definition of Lie groups, classical Lie groups.
  1.2 Coverings of Lie groups
  1.3 The Lie algebras of Lie groups
  1.4 Representation of Lie groups, tensor products, exterior products, symmetric products.
  1.5 The exponential map
  1.6 Solvable Lie algebras, nilpotent Lie algebras, semi-simple Lie algebras.
  1.7 Engel's theorem, Lie's theorem
  1.8 Complete reducibility of representations of semi-simple Lie algebras, Weyl's unitarian trick.
  1.9 Universal enveloping algebra and the Poincare-Birkhoff-Witt theorem

2. Classical Lie algebras and their representations
  2.1 sl(2) and its representations
  2.2 sl(n): roots system, representations, root lattice, weight lattice, the dominant Weyl chamber, Weyl group.

2.3 sp(2n), so(2n), so(2n+1): roots system, representation, root lattice, weight lattice, the dominant Weyl chamber, Weyl group

3. Semi-simple Lie algebras
   3.1 Cartan's criterion for a Lie algebra to be solvable.
   3.2 Killing form, properties of the Killing form for a semisimple Lie algebra,
   3.3 Casmir operator, the complete reducibility of representations of semi-simple Lie algebras (an algebraic proof), Jordan decomposition of elements in a complete Lie algebra
   3.4 Root systems and structure for sem-simple Lie algebras
   3.5 Representations of semisimple Lie algebras
   3.6 Weyl group, Weyl chambers of a semsimple Lie algebra, the highest weight of a representation
   3.7 Dykin diagram
   3.8 Classification of semisimple Lie algebras
4. Weyl character formula
   4.1. The Grothendieck ring of a semisimple Lie algebra, characters of representations
   4.2 Compact form, maximal compact subgroup of a semisimple Lie group, maximal torus
   4.3 Weyl character formula and dimension formula
   4.4 Weyl integration formula, proof of the Weyl character formula (optional)

Geometric representation theory 1

3 credits

Prerequisite: Representation Theory, Lie groups and Lie algebras

Reference:
Humphreys - Representations of semisimple Lie algebras in the BGG category O
Hotta, Takeuchi & Tanisaki - D-modules, perverse sheaves, and representation theory.

Contents
Chapter 1. Category O
- 1.1 Verma modules and category O
- 1.2 Harish–Chandra isomorphism
- 1.3 Blocks of category O and linkage principle
- 1.4 Proof of Weyl character formula using Verma modules
- 1.5 Projective modules, standard filtrations and BGG reciprocity
- 1.6 Translation functors
- 1.7 Highest weight categories

Chapter 2. Kazhdan–Lusztig conjecture

Geometric representation theory 2

3 credits

Prerequisite: Representation theory, Lie groups and Lie algebras.

Textbook: Chris, Ginzburg – Representation theory and complex geometry.

Reference: Achar - Perverse sheaves and applications to representation theory.

Contents

3.4 Sheaves on the nilpotent cone
*3.5 Generalized Springer correspondence

*Chapter 4. Character sheaves and microlocal geometry
*4.1 Horocycle transform
*4.2 Induction and Restriction
*4.3 Introduction to micro-support of sheaves
*4.4 Microlocal description of perverse sheaves
*4.5 Microlocal description of character sheaves

*Chapter 5. Hecke algebra and K-theory
*5.1 Affine Weyl group and Hecke algebra
*5.2 K-theory and equivariant K-theory
*5.3 Steinberg variety
*5.4 K-theoretic description of affine Hecke algebra

*: Sections with * are up to the lecturer's choice.