Classical Mechenics (25 pts)

Consider the two dimensional harmonic oscillator with the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2). \tag{1}$$

- Write down the Euler-Lagrangian equations of the system, and find their general solutions.
- Show that the following transformation

  $$x \mapsto x\cos\theta - y\sin\theta, \quad y \mapsto x\sin\theta + y\cos\theta \tag{2}$$

  is a symmetry of the system. Write down the corresponding conserved charge using the Noether theorem.
- Find the expressions for generalized momenta p_x and p_y corresponding to x and y. Then write down the Hamiltonian of the system as a function of generalized coordinates and momenta.
- Define

  $$H_x = \frac{p_x^2}{2m} + \frac{k}{2}x^2, \quad H_y = \frac{p_y^2}{2m} + \frac{k}{2}y^2. \tag{3}$$

  Show that both H_x and H_y are conserved quantities.
- Write down the Hamilton-Jacobi (HJ) equation of the system. Find the general solution of the HJ equation.

Quantum Mechanics (25 pts)

Consider the Hamiltonian of a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{4}$$

Here H, p, x should be understood as Hermitian operators. Define

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right), \tag{5}$$

and

$$N = a^\dagger a. \tag{6}$$

- Using the canonical commutation relation between x and p, compute

  $$[a, a^\dagger], \quad [N, a^\dagger], \quad [N, a]. \tag{7}$$

  You need to provide all necessary details of your derivation.

- Express H in terms of a, $a^\dagger$, and/or N. You need to provide all necessary details of your derivation.
- Using a and $a^\dagger$, give a definition of the vacuum state $|0\rangle$, then work out all eigenstates of H.
- Compute $\langle 0|x|0\rangle$, $\langle 0|p|0\rangle$, $\langle 0|x^2|0\rangle$, $\langle 0|p^2|0\rangle$.

Thermodynamics (25 pts)

A system contains N independent sites and N electrons at temperature T. At a given site there is one accessible orbit, but that orbit can be empty, occupied by one electron of either spin, or occupied by two electrons of opposite spin. The electrons can hop from site to site.

The site energy is zero if the site is either empty or singly occupied, and Δ if it is doubly occupied.

- Determine the grand canonical partition function of a single site.
- Determine the grand canonical partition function of N independent sites.
- Determine the chemical potential as a function of temperature of the system.
- Compute the mean energy and heat capacity of the system.

Electrodynamics (25 pts)

Consider a spherical shell of radius R with charge Q uniformly distributed on its surface, which spins around an axis with constant angular velocity $\vec{\omega}$.

- Compute the current density.
- Find the vector potential inside and outside the sphere.
  Hint:
  $$\int_{-1}^{1} \frac{\xi d\xi}{(1-\alpha\xi)^{1/2}} = \frac{2}{3\alpha^2}\left((2-\alpha)\sqrt{1+\alpha} - (2+\alpha)\sqrt{1-\alpha}\right).$$
- Find the magnetic field inside and outside the sphere.
  Hint:
  $$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}).$$
- Find the magnetic dipole moment from the view of an observer outside the sphere.
  Hint: the magnetic field of a dipole can be written in the following form
  $$\mathbf{B}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right),$$
  where $\mathbf{m}$ is the magnetic dipole moment.