清华大学考试试题专用纸

姓名: ____________ 学号: ____________

Everyone need to solve Problem $1, 2, 3, 4$ (10 points each). Please select four problems in the following table and solve them (15 points each).

Problem 5	Problem 6	Problem 7	Problem 8	Problem 9	Problem 10

Notations: $\mathbf{S}^n$ means the sphere of dimension n. $\mathbf{RP}^n$ means the real projective space of dimension n. $\mathbf{CP}^n$ means the complex projective space of **complex** dimension n.

1. Solve the following problems.

   (A) What's the fundamental group of SO(3)?

   (B) Does SO(3) admit a smooth Riemannian metric with constant Ricci curvature?

2. Solve the following problems.

   (A) Is it true that every continuous map $f : \mathbf{S}^{2024} \to \mathbf{RP}^{2024}$ is null-homotopic?

   (B) Is it true that every continuous map $g : \mathbf{S}^{2024} \to \mathbf{CP}^{1012}$ is null-homotopic?

3. Let (M, g) be a connected Riemannian manifold of dimension n. Suppose that there exists some $f \in C^\infty(M, \mathbf{R})$ such that

$$\mathrm{Ric}(g) = (n-1)fg.$$

   (A) If $n = 2$, is f necessarily a constant?

   (B) If $n = 2024$, is f necessarily a constant?

4. Consider the quotient space $X = ([0,1] \times \mathbf{S}^1 \times \mathbf{S}^1)/\sim$, where the equivalence relation $\sim$ is generated by

$$(0, x, y) \sim (0, z, w) \quad \text{if} \quad xy = zw,$$

   and

$$(1, x, y) \sim (1, z, w) \quad \text{if} \quad x^2y^6 = z^2w^6.$$

   Here we treat $\mathbf{S}^1$ as the space of unit complex numbers. Compute $H_n(X; \mathbf{Z})$ for all n.

5. Recall a space X is called an **H-space** if there exists a point $e \in X$ and a continuous map $\mu : X \times X \to X$ such that the map

$$X \to X \text{ defined by } x \mapsto \mu(e, x)$$

and the map

$$X \to X \text{ defined by } x \mapsto \mu(x, e)$$

are both homotopic to the identity map. Show that $\mathbf{CP}^n$ is **not** an H-space for any $1 \leq n < \infty$.

6. Let (M, g) be a Cartan-Hadamard manifold. Given $p \in M$, let

$$f : M \to [0, +\infty)$$

be the function $f(x) = \frac{1}{2} d(x, p)^2$. Show that f is strictly geodesically convex, i.e. for any (nontrivial) geodesic $\gamma : [0, 1] \to M$, the following inequality holds for all $t \in (0, 1)$

$$f(\gamma(t)) < (1 - t) f(\gamma(0)) + t f(\gamma(1)).$$

7. Let M be an oriented, connected, closed manifold of dimension $n \geq 2$. Let $f : \mathbf{S}^n \to M$ be a continuous map of mapping degree $\deg(f) = 1$. Show that f must be a homotopy equivalence.

8. Let (M, g) be a smooth Riemannian manifold of dimension n and $p \in M$. Show that when r is small enough,

$$\text{Vol}(B(p, r)) = \omega_n r^n \left(1 - \frac{s(p)}{6(n + 2)} r^2 + O(r^3)\right)$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$ and $s(p)$ is the scalar curvature of (M, g) at point p.

9. Let M be a smooth, closed manifold of dimension ≥ 1. And let $f : M \to M$ be a smooth map such that $f \circ f(x) = x$ for any $x \in M$. Show that the set $\{x \in M \mid f(x) = x\}$ can **not** be a single point.

10. Solve the following problems.

    (A) Does $\mathbf{S}^1 \times \mathbf{S}^1$ admit a Riemannian metric with conjugate radius $= +\infty$?

    (B) Does $\mathbf{S}^1 \times \mathbf{S}^1$ admit a Riemannian metric with conjugate radius $< +\infty$?

    (C) Does $\mathbf{S}^2 \times \mathbf{S}^1$ admit a Riemannian metric with conjugate radius $= +\infty$?