Qualifying Exam: 2024 Fall

考试课程: Probability & Statistics　　姓名: ___________　　学号: ___________

- There are 11 problems in this exam (4 pages). You need to choose 8 of them to solve. If you select more than 8, only the first 8 that you have worked on will be graded. Note that 4 of the problems are worth 15 points each and the rest 10 points each.
- You must follow all the rules of exam taking. Misconducts will be subject to proper disciplinary actions by the Center.
- You must provide all necessary details for full credits. A final answer with no or little explanation/derivation, even if correct, receives a minimal credit.
- $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of positive integers. $\xrightarrow{(d)}$ and $\overset{(d)}{=}$ mean "converges in distribution" and "equal in distribution", respectively.

1. (10 points) Let $U_1, U_2, \dots$ be independent identically distributed (i.i.d.) random variables uniformly distributed on $[0, 1]$, and define $S_n = \sum_{k=1}^{n} U_k$.

   (a) Calculate $\mathbb{E}[(S_n)^4]$.

   (b) Determine the distribution of S_3.

2. (10 points) Let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables such that

$$\sqrt{n}(Y_n - a) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, \sigma^2),$$

   where $\mathcal{N}(0, \sigma^2)$ stands for the normal distribution with $\sigma \neq 0$.

   (a) Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that it is differentiable at a and $g'(a) \neq 0$. Prove that

$$\sqrt{n}(g(Y_n) - g(a)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, g'(a)^2\sigma^2\right).$$

   (b) Fix $p \in (0, 1)$. For $n \in \mathbb{N}$, let $Z_n \overset{(d)}{=} \text{Bin}(n, p)$ be a binomial random variable. Prove that

$$\sqrt{n}\left(\ln\left(\frac{Z_n}{n}\right) - \ln p\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, \frac{1-p}{p}\right).$$

3. (10 points) Let X be a real-valued random variable. Prove that the following properties are equivalent in the sense that the parameters $K_i > 0$ appearing in these properties differ from each other by at most an absolute constant factor.

   (a) The tails of X satisfy that

$$\mathbb{P}(|X| \geq t) \leq 2\exp\left(-t/K_1\right) \quad \text{for all } t \geq 0.$$

   (b) The moments of X satisfy that

$$\mathbb{E}\left(|X|^p\right) \leq (K_2 p)^p \quad \text{for all } p \in \mathbb{N}.$$

(c) The moment generating function of $|X|$ is bounded at some point, i.e.,

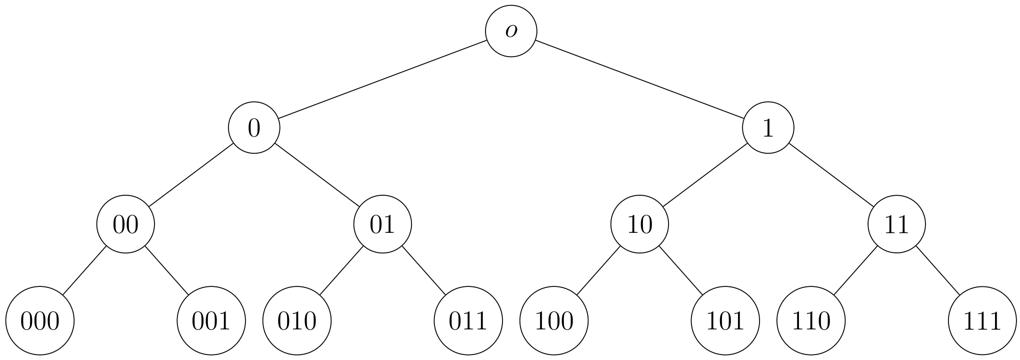
$$\mathbb{E}\exp(|X|/K_3) \le 2$$

for some $K_3 > 0$.

Hint: You can use Stirling's approximations: $e(n/e)^n \le n! \le en(n/e)^n$ for all $n \in \mathbb{N}$ and

$$n! = (1 + o(1))\sqrt{2\pi n}(n/e)^n \quad \text{for large } n.$$

4. (10 points) Let $\mathbb{T}_d$ be an infinite d-regular tree, where every node has degree d. On the other hand, let $\mathcal{T}_b$ be an infinite b-ary tree with root o. In other words, $\mathcal{T}_b$ is an infinite tree where every node has b children nodes and every non-root node has one parent node. Below is an illustration of a binary tree with $b = 2$.



Consider the simple random walk X_n on $\mathbb{T}_d$ and simple random walk Y_n on $\mathcal{T}_b$, where at each step, the walker moves to the neighbor nodes with equal probability.

(a) For each $d \in \mathbb{N}$ with $d \ge 2$, determine whether the simple random walk X_n on $\mathbb{T}_d$ is recurrent or transient. Prove your claim.

(b) For each $b \in \mathbb{N}$, determine whether the simple random walk Y_n on $\mathcal{T}_b$ is recurrent or transient. Prove your claim. (Hint: Use part (a).)

5. (15 points) Let $X_1, X_2, \ldots$ be i.i.d. random variables with exponential distribution: $\mathbb{P}(X_k > x) = e^{-x}$ for $x \ge 0$. Define

$$M_n := \sum_{k=1}^{n} \frac{X_k}{k}.$$

(a) Prove that $(M_n - \ln n)_{n\in\mathbb{N}}$ converges to a limit Y almost surely.

(b) Prove that, for every $p \in (0, 1)$, $\left(\frac{\exp(pM_n)}{n^p}\right)_{n\in\mathbb{N}}$ converges to a limit Z in L^1.

6. (15 points) Let B_t be a one-dimensional (1D) standard Brownian motion started from 0.

(a) Consider a Brownian motion $X_t = x + B_t$ started at some $x > 0$. For any $t > 0$ and $b > a > 0$, compute the probability of $X_t \in [a, b]$ conditioning on that X_t does not hit zero between 0 and t, i.e.,

$$\mathbb{P}\left(X_t \in [a, b] \,\middle|\, \min_{0\le s\le t} X_s > 0\right).$$

(b) A **Brownian bridge** W_t on $[0,1]$ is a 1D standard Brownian motion B_t subject to the condition that $B_1 = 0$. In other words, $W_t = (B_t | B_1 = 0)$ is a continuous-time Gaussian process whose probability distribution is the conditional probability distribution of B_t conditioning on $B_1 = 0$. A 1D **Gaussian free field** (GFF) h_t on $[0,1]$ with zero boundary is a continuous-time Gaussian process subject to the zero boundary condition $h_0 = h_1 = 0$ and has zero mean $\mathbb{E}h_t = 0$, $t \in [0,1]$, and covariances

$$\mathbb{E}(h_t h_s) = G(t,s), \quad t, s \in [0,1].$$

Here, $G(t,s)$ is the Green's function of the Laplace operator $-\Delta$, i.e., $G(t,s)$ is the unique continuous function such that for any smooth test function $f \in C_c^\infty(0,1)$,

$$\int_0^1 G(t,s) \frac{\partial^2}{\partial t^2} f(t) \mathrm{d}t = -f(s) \quad \text{and} \quad G(0,s) = G(1,s) = 0.$$

Prove that the process $(W_t : t \in [0,1])$ has the same distribution as the process $(h_t : t \in [0,1])$ in the sense that for any fixed $0 \le t_1 < t_2 < \dots t_n \le 1$ and Borel sets $O_1, O_2, \dots, O_n$,

$$\mathbb{P}(W_{t_1} \in O_1, \dots, W_{t_n} \in O_n) = \mathbb{P}(h_{t_1} \in O_1, \dots, h_{t_n} \in O_n).$$

(Hint: Find the explicit form of the function $G(t,s)$ and calculate $\mathbb{E}(W_t W_s)$.)

7. (10 points) Let $X_1, ..., X_n$ be an iid sample from $\mathrm{N}(\mu, 1)$ with μ unknown. Unfortunately, one forgets to record $X_1, ..., X_n$ in a study and only records $\mathbf{Y} = (Y_1, ..., Y_n)$ where $Y_i = I(X_i < 0)$ and $I(\cdot)$ is the indicator function.

   (a) Derive the MLE of μ based on the observed data $\mathbf{Y}$.

   (b) Construct a **size** α uniformly most powerful (UMP) test for testing $H_0 : \mu \le \mu_0$ versus $H_1 : \mu > \mu_0$ based on the observed data $\mathbf{Y}$.

   (c) Describe how to construct a $(1-\alpha)$ confidence interval for μ based on the observed data $\mathbf{Y}$.

8. (10 points) Consider the following linear model $Y_i = \mathbf{z}_i^T \beta + \epsilon_i$, $i = 1, ..., n$. $\mathbf{z}_1, ..., \mathbf{z}_n \in R^d$ are fixed and given, and $\beta \in R^d$ is unknown. $\epsilon_i' s$ are random variables satisfying the Gauss-Markov assumptions that $\mathrm{E}[\epsilon_i] = 0$, $\mathrm{Var}[\epsilon_i] = \sigma^2$ and $\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0, \forall i \ne j$. Let $\mathbf{Y} = (Y_1, ..., Y_n)^T$, and $\mathbf{Z} = \begin{pmatrix} \mathbf{z}_1^T \\ \mathbf{z}_2^T \\ \vdots \\ \mathbf{z}_n^T \end{pmatrix}$ be the n by d design matrix.

   (a) Let $\hat{\beta}$ be the least squares estimate of β which is given by $\hat{\beta} = (\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \mathbf{Y}$. Let $\theta = \mathbf{b}^T \beta$ where $\mathbf{b} \in R^d$ is a known vector. **Write down** the mean and the variance of $\hat{\theta}$ where $\hat{\theta} = \mathbf{b}^T \hat{\beta}$. Further, **prove** that under the Gauss-Markov assumptions, the estimator $\hat{\theta}$ has the smallest variance among all linear unbiased estimator of θ. Here linear unbiased estimator we mean estimator in the form of $\mathbf{c}^T \mathbf{Y}$ and is unbiased for θ.

   (b) Further assume that $(\epsilon_1, ..., \epsilon_n)$ are iid from $\mathrm{N}(0, \sigma^2)$ with σ^2 known. Derive the information matrix $I(\beta)$.

9. (10 points) Suppose $X_1, ..., X_n$ are IID from the uniform distribution on $[0, \theta]$ for some unknown $\theta > 0$. Fix $t \in (0, \theta)$. Consider two estimators of $P(X_1 \le t)$: $F_n(t) = \frac{1}{n} \sum_{i=1}^n I_{\{X_i \le t\}}$ and $T_n(t) = t/(2\bar{X})$, where $\bar{X}$ is the sample mean.

   (a) Find the asymptotic distributions of the two estimators.

(b) For what value of t will the first estimator have a smaller asymptotic variance than the second estimator?

(c) Let $\theta = 1$. For the $F_n(t)$ defined above, find the asymptotic distribution of $nF_n(n^{-1/2}) - \sqrt{n}$.

10. (15 points) Let $X_1, ..., X_n$ ($n \geq 2$) be iid from $N(\mu, \sigma^2)$ distribution with $\mu \geq 0$ and $\sigma > 0$ being the unknown parameters. Let $\bar{X}$ and S^2 be the sample mean and sample variance, respectively. Recall χ_k^2 has probability density function

$$\frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1}e^{-\frac{x}{2}}, x \geq 0.$$

(a) Show $\bar{X}$ and S^2 are independent.

(b) Find UMVUE of μ/σ if it exists.

(c) Is $\bar{X}$ admissible for estimating μ under the square error loss? Prove your assertion.

11. (15 points) Let $X_1, ..., X_n$ be an iid sample from Uniform$[\theta,\ \theta + |\theta|]$ where $\theta \neq 0$.

(a) Derive the method of moments estimator of θ

(b) Derive the MLE of θ, $\hat{\theta}$.

(c) Is $\hat{\theta}$ a consistent estimator of θ? Please explain your answer.