Note: we use Einstein summation convention here. Repeated indices means summation over all possible values of indices unless otherwise stated. The speed of light c and the gravitational constant G is taken to be 1. Without further specialization, we are working on four dimensional manifold. **There are two pages in the exam!**

Einstein metric

Consider the vacuum Einstein's equation with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 0. \tag{1}$$

1. (10) Proof that $R_{\mu\nu} = k g_{\mu\nu}$ where k is a constant and find out the value of k.

Now start with an ansatz of a metric in the following form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

where $f(r)$ is a **polynomial** in r.

2. (10) Compute the Christoffel symbols $\Gamma^{\mu}{}_{\nu\sigma}$ for this metric.
3. (10) Compute the Ricci tensor $R_{\mu\nu}$ and scalar curvature R of this metric.
4. (10) Assuming that the above ansatz is a solution of the vacuum Einstein equation with cosmological constant Λ, solve $f(r)$.
5. (10) For $\Lambda > 0$ and $\Lambda < 0$, write down all the Killing vectors of the metric using the solution you found.

You may find the following formulae useful

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\rho\lambda}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\lambda}}{\partial x^{\rho}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\lambda}} \right), \tag{3}$$

and

$$R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\sigma}{}_{\mu\rho} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\sigma}{}_{\nu\rho} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\sigma}{}_{\lambda\mu}. \tag{4}$$

the $\lambda\phi^3$ model

Consider the following Lagrangian density of a real scalar $\phi(x)$ in $3+1$ dimensional spacetime with flat metric $(-,+,+,+)$,

$$\mathcal{L} = -\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\phi^3, \tag{5}$$

where g is a coupling with dimensions of mass.

1. (10) draw all the one-particle irreducible Feynman diagram up to 3 external lines and 2 loops.

2. (10) Compute the one-loop self-energy graph using dimensional regularization.

3. (10) Introducing $m^2 = m_R^2 + \delta m^2$. What is the value of δm^2 if we want to write the one-loop self-energy graph as a finite function of m_R?

4. (10) Compute the one-loop tadpole graph using dimensional regularization.

5. (10) What counter-term should you introduce in the Lagrangian to cancel the one-loop tadpole diagram?

You may find the following formula useful

$$(AB)^{-1} = \int_0^1 dx[xA + (1-x)B]^{-2}, \tag{6}$$

$$\int d^d k \frac{1}{(-k^2 - 2p \cdot k - M^2 + i\epsilon)^s} = (-1)^s i\pi^{d/2} \frac{\Gamma(s - d/2)}{\Gamma(s)} (-p^2 + M^2 - i\epsilon)^{d/2-s}, \tag{7}$$

where $\Gamma(z)$ is the Gamma function which has a simple pole at the origin.

$$\Gamma(z) = \frac{1}{z} + \gamma + \mathcal{O}(z), \tag{8}$$

where γ is the Euler constant.